Solutions to Homework 2

Math 320

April 12, 2019

2.2.  (a) The characteristic polynomial of \[
\begin{pmatrix}
1 & 2 \\
0 & 3
\end{pmatrix}
\] is \(\lambda^2 - 4\lambda + 3\), so the eigenvalues are 1 and 3. The corresponding eigenvectors are \(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\) and \(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\). The general solution is
\[
X(t) = \alpha e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

(b) The characteristic polynomial of \[
\begin{pmatrix}
1 & 2 \\
3 & 6
\end{pmatrix}
\] is \(\lambda^2 - 7\lambda\), so the eigenvalues are 0 and 7. The corresponding eigenvectors are \(\begin{pmatrix} 2 \\ -1 \end{pmatrix}\) and \(\begin{pmatrix} 1 \\ 3 \end{pmatrix}\). The general solution is
\[
X(t) = \alpha \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \beta e^{7t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}.
\]

(c) The characteristic polynomial of \[
\begin{pmatrix}
1 & 2 \\
1 & 0
\end{pmatrix}
\] is \(\lambda^2 - \lambda - 2\), so the eigenvalues are 2 and -1. The corresponding eigenvectors are \(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\) and \(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\). The general solution is
\[
X(t) = \alpha e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \beta e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

(d) The characteristic polynomial of \[
\begin{pmatrix}
1 & 2 \\
3 & -3
\end{pmatrix}
\] is \(\lambda^2 + 2\lambda - 9\), so the eigenvalues are \(-1 \pm \sqrt{10}\). The corresponding eigenvectors are \(\begin{pmatrix} 2 \\ -2 \pm \sqrt{10} \end{pmatrix}\). The general solution is
\[
X(t) = \alpha e^{(-1+\sqrt{10})t} \begin{pmatrix} 2 \\ -2 + \sqrt{10} \end{pmatrix} + \beta e^{(-1-\sqrt{10})t} \begin{pmatrix} 2 \\ -2 - \sqrt{10} \end{pmatrix}.
\]
2.3. In each case we know the eigenvalues and eigenvectors of the matrix, we ask which picture has a straight-line solution along the line spanned by each eigenvector, and we check that the straight-line solution flows away from the origin, flows toward the origin, or is an equilibrium solution according as the eigenvalue is positive, negative, or zero.

(a) 4. (b) 2. (c) 1. (d) 3.

2.6. We set \( y = x' \), so the equation becomes
\[
\begin{align*}
  x' &= 0x + 1y \\
  y' &= -by - kx
\end{align*}
\]
or
\[
X' = \begin{pmatrix} 0 & 1 \\ -b & -k \end{pmatrix} X.
\]
The characteristic polynomial is \( \lambda^2 + k\lambda + b \), which has distinct real roots if and only if \( k^2 > 4b \). In that case the eigenvalues are \( \frac{-k \pm \sqrt{k^2 - 4b}}{2} \). We remark that both of these are negative, so any solution will decay asymptotically to 0 as \( t \to \infty \). The corresponding eigenvectors are \( \begin{pmatrix} 1 \\ -k \pm \sqrt{k^2 - 4b} \end{pmatrix} \). The general solution is
\[
X(t) = \alpha e^{(-k + \sqrt{k^2 - 4b})t/2} \begin{pmatrix} 1 \\ -k \pm \sqrt{k^2 - 4b} \end{pmatrix} + \beta e^{(-k - \sqrt{k^2 - 4b})t/2} \begin{pmatrix} 1 \\ -k - \sqrt{k^2 - 4b} \end{pmatrix},
\]
and we really we only care about the first component \( x(t) \):
\[
x(t) = \alpha e^{(-k + \sqrt{k^2 - 4b})t/2} + \beta e^{(-k - \sqrt{k^2 - 4b})t/2}.
\]
If \( x(0) = 0 \) then we find that \( \alpha + \beta = 0 \), so
\[
x(t) = \alpha \left( e^{(-k + \sqrt{k^2 - 4b})t/2} - e^{(-k - \sqrt{k^2 - 4b})t/2} \right),
\]
or if you like hyperbolic sines and cosines,
\[
x(t) = 2\alpha e^{-kt/2} \sinh \left( \sqrt{k^2 - 4b} \frac{t}{2} \right).
\]
If \( x'(0) = 1 \) then we find that \( \alpha = 1/\sqrt{k^2 - 4b} \):

\[
x(t) = \frac{e^{(-k+\sqrt{k^2-4b})t/2} - e^{(-k-\sqrt{k^2-4b})t/2}}{\sqrt{k^2 - 4b}}
\]
or

\[
x(t) = \frac{e^{-kt/2} \sinh\left(\sqrt{k^2 - 4b} \frac{t}{2}\right)}{\sqrt{k^2 - 4b}/2}.
\]

Physically, the mass starts at 0 with an upward velocity of 1, goes up for a bit, then comes asymptotically back to 0.

2.7. The eigenvalues of \( \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix} \) are \( a \) and 1, so \( A \) has repeated real eigenvalues when \( a = 1 \). When \( a \neq 1 \), the eigenvector for \( a \) is \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), and the eigenvector for 1 is \( \begin{pmatrix} 1 \\ -a + 1 \end{pmatrix} \). As \( a \to 1 \), the latter approaches \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \), that is, the two eigenvectors come together.

2.8. Let

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]

and let \( \lambda \) and \( \mu \) be the eigenvalues of \( A \). Then the characteristic polynomial of \( A \) is on the one hand

\[
t^2 - (a + d)t + (ad - bc)
\]
and on the other hand

\[
(t - \lambda)(t - \mu),
\]
so we have

\[
\lambda + \mu = a + d \quad \lambda \mu = ad - bc.
\]
(continued on next page)
Now for the first column of $A - \lambda I$ we have

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a - \lambda \\ c \end{pmatrix} = \begin{pmatrix} a^2 - a\lambda + bc \\ ac - c\lambda + cd \end{pmatrix}
\]

\[
= \begin{pmatrix} a(a + d - \lambda) - ad + bc \\ (a + d - \lambda)c \end{pmatrix}
\]

\[
= \begin{pmatrix} a\mu - \lambda\mu \\ \mu c \end{pmatrix}
\]

\[
= \mu \begin{pmatrix} a - \lambda \\ c \end{pmatrix},
\]

and for the second column of $A - \lambda I$ it is similar.

Alternatively, if you know the Cayley–Hamilton theorem, you could argue as follows. Every matrix is a root of its own characteristic polynomial: $(A - \mu I)(A - \lambda I) = 0$. Thus the columns $V$ of $A - \lambda I$ satisfy $(A - \mu I)V = 0$, so they are $\mu$-eigenvectors of $A$. 

4