Last time: an ordering on a field F

is a subset $F^+ \subset F$
such that $\forall a, b \in F^+$ $a+b, ab \in F^+$
and $\forall a \in F'$, either $a \in F^{+'}$
or $-a \in F^+$
or $a = 0$.

define $a < b$ to mean $b - a \in F^+$

our favorite field $\mathbb{Q}$ has only one reasonable ordering: $\mathbb{Q}^+ = \left\{ a/b \;\middle|\; \begin{array}{l} a>0 \text{ and } b>0 \\ \text{or} \\ a<0 \text{ and } b<0 \end{array} \right\}$

worksheet: $\mathbb{R}(x) = \left\{ \frac{p(x)}{q(x)} \right\} = \left\{ \frac{a_m x^m + \cdots}{b_n x^n + \cdots} \right\}$

get an ordering by saying

$$\left\{ \frac{a_m x^m + \cdots}{b_n x^n + \cdots} > 0 \text{ iff } \frac{a_m}{b_n} > 0 \right.$$

then $x >$ every number $1, 2, 3, 4, \ldots$

because $x - 4 = \frac{1 \cdot x - 4}{1}$ $\quad \frac{1}{1} > 0$

and $\frac{1}{x} <$ every number

$$\frac{1}{x} - 4 = \frac{1 - 4x}{x} = \frac{-4x + 1}{1 \cdot x} \qquad \frac{-4}{1} < 0$$

last prob. on worksheet:
could have done the opposite
$0 < x <$ every const. polynomial
$1/x >$ every const. polynomial

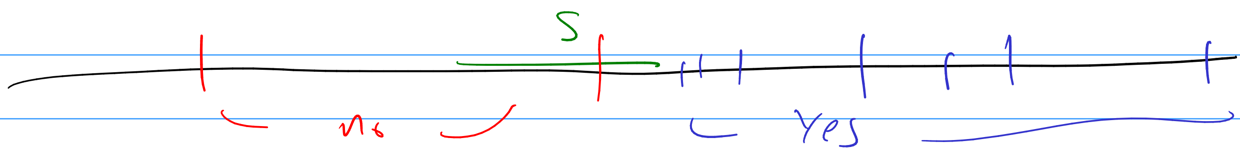
Let X be any ordered set.
(favorite example: $\mathbb{Q}$
or subsets of $\mathbb{Q}$
with usual ordering.)

an <u>upper bound</u> for a subset

$S \subset X$ is an element $x \in X$

such that $\forall s \in S$, $s \leq x$

x



lub vs sup.

a lower bound is an $x \in X$
s.t. $\forall s \in S$, $x \leq s$.

or supremum

x is the least upper bound for $S \subset X$
if ① it's an upper bound, and

② $\forall$ other upper bound y, we have

$x \leq y$.

similarly, the greatest lower bound or infimum

glb vs inf

use whatever terms you prefer.

examples on worksheet.

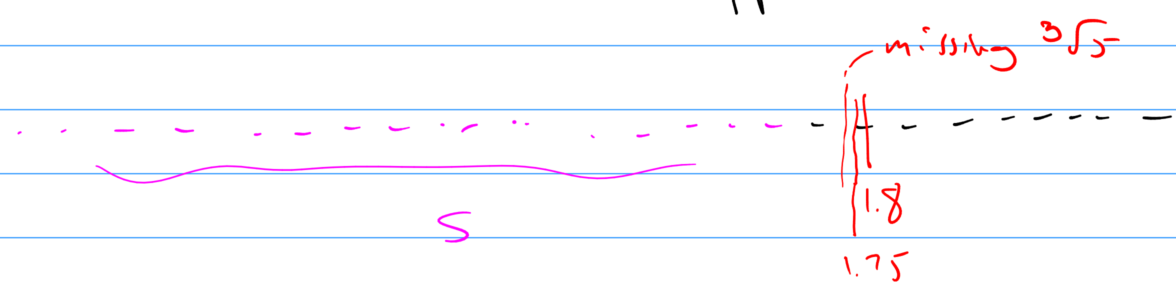
X has the least upper bound property

if every subset $S \subset X$ with an upper bound has a least upper bound (in X)

$\mathbb{Q}$ does not have the l.u.b. property:

$$S = \{ s \in \mathbb{Q} \mid s^3 \leq 5 \}$$

has an upper bound, but no least upper bound.



Theorem $\exists$ an ordered field $\mathbb{R}$ containing a copy of $\mathbb{Q}$

that has the l.u.b. property

any other such field is isomorphic to $\mathbb{R}$

The Real Numbers

low-brow model: (decimals)

$\mathbb{R}$ = symbols like 123.456789...

$A_n A_{n-1} \dots A_1 A_0 . a_0 a_1 a_2 a_3 \dots$

where $A_i, a_j \in \{0, 1, 2, \dots, 9\}$

doesn't end in 9999....

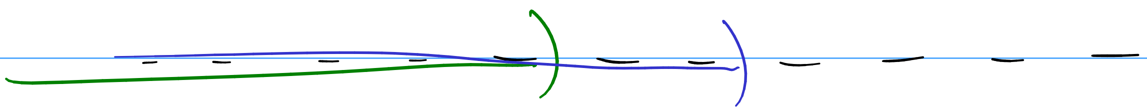
addition + multiplication – can define them but it's a pain.

middle-brow model (Dedekind cuts)

$\mathbb{R}$ = things like $\{s \in \mathbb{Q} \mid s^3 < 5\}$

or $\{s \in \mathbb{Q} \mid s < 12/7\}$

subsets $S \subset \mathbb{Q}$ | if $s \in S$ and $t < s$ then $t \in S$.



embed $\mathbb{Q} \hookrightarrow \mathbb{R}$

$9/6 \longmapsto \{s \in \mathbb{Q} \mid s < 9/6\}$

addition: $S + T = \{s + t \mid s \in S, t \in T\}$

ordering: $S \leq T$ means $S \subseteq T$

multiplication: 4 cases depending on whether $S > 0$ or $S < 0$
$T > 0$ or $T < 0$.

high-brow model: Cauchy sequences.

$\mathbb{R}$ = things like the seq.

$3, 3.1, 3.14, 3.141, \ldots \quad (\to \pi)$

Same... $\left(\begin{array}{l} 1/2, 3/4, 7/8, 15/16, \ldots \quad (\to 1) \\ 1, 1, 1, 1 \ldots \quad (\to 1) \end{array} \right.$

Sequences $a_1, a_2, \ldots \in \mathbb{Q}$

s.t. $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ such that
$m, n \geq N \Rightarrow |a_m - a_n| < \varepsilon$.

up to this equivalence relation:

$a_i \sim b_j$ if $\lim_{n \to \infty} (a_n - b_n) = 0$

for $a_1, a_2 \ldots$
$b_1, b_2 \ldots$

define sum = $a_1 + b_1, a_2 + b_2, \ldots$

work: check this is well-defined.