Complex Numbers

$\mathbb{C}$ = take $\mathbb{R}$ and throw in a new number i ("imaginary") such that $i^2 = -1$

examples: $1+2i$ $\qquad$ $2-i$

in general, $\mathbb{C} = \{ a+bi \mid a, b \in \mathbb{R} \}$

add: $(a+bi) + (c+di)$

$$= (a+c) + (b+d)i$$

multiply: $(a+bi)(c+di) = ac + adi + bci + bd\underbrace{i^2}_{=-1}$

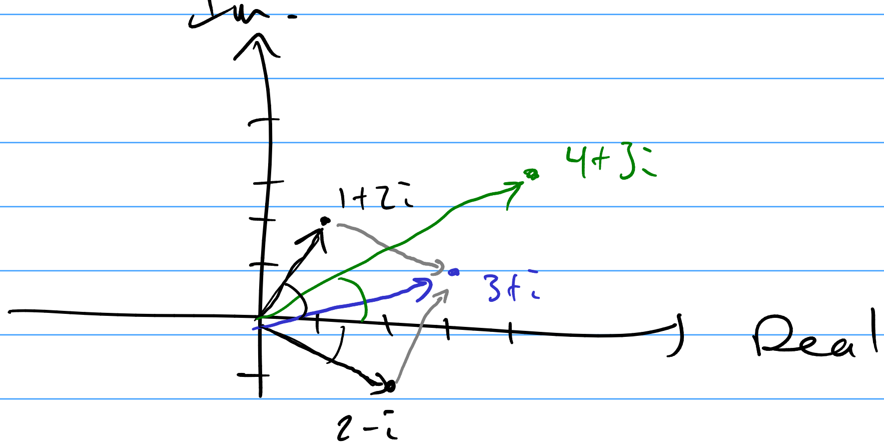
$$= (ac-bd) + (ad+bc)i$$

example: $(1+2i) + (2-i) = \underline{3+i}$

$$(1+2i)\cdot(2-i) = 2 - i + 4i \cancel{-2i^2} + 2$$

$$= \underline{4+3i}$$

Draw $\mathbb{C}$ as the plane

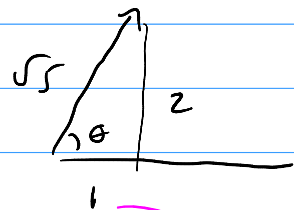


Geom. meaning of addition?
Just like vectors: draw them head to tail...

Geom. meaning of multiplication?
becomes clear in polar coords

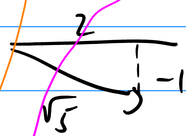
$$(1+2i)(2-i) = 4+3i$$

$1+2i \rightsquigarrow$ length is $\sqrt{1^2+2^2} = \sqrt{5}$



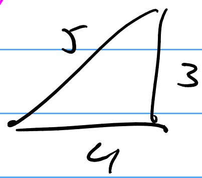
angle $\cos\theta = 1/\sqrt{5}$ so $\theta \approx 63°$

$2-i \rightsquigarrow$ length is $\sqrt{5}$



angle sat.
$\cos\theta = 2/\sqrt{5}$ so $\theta \approx -27°$

$4+3i \rightsquigarrow$ length is $\sqrt{4^2+3^2} = 5$
$\cos\theta = 4/5$ so $\theta \approx 37°$



multiply the lengths
add the angles!

Why does it work?

write $a+bi = r\cos\theta + i\,r\sin\theta$

$c+di = s\cos\phi + i\,s\sin\phi$

multiply them out:

$$rs\cos\theta\cos\phi - rs\sin\theta\sin\phi$$

$$+\, irs\cos\theta\sin\phi + irs\sin\theta\cos\phi$$

$$= rs\cos(\theta+\phi) + irs\sin(\theta+\phi)$$

this def. of $+$ and $\cdot$ makes $\mathbb{C}$ a commutative ring.

in fact it's a field!

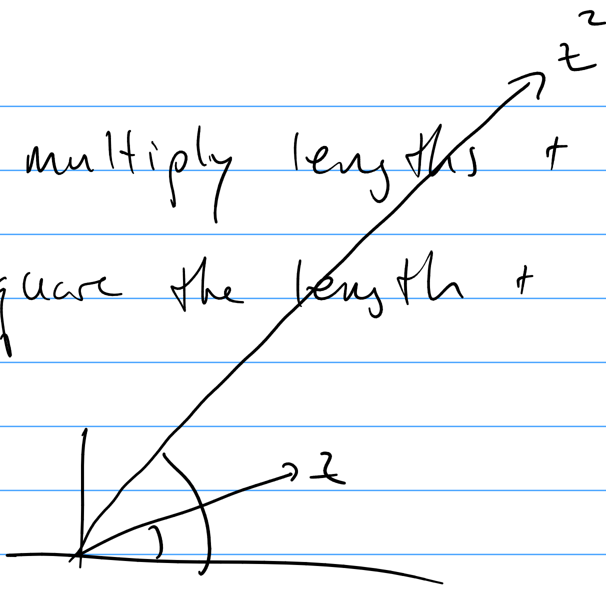
$\dfrac{1}{1+2i}$ is not obviously of the form $\underline{\underline{a+bi}}$

but $$\frac{1}{1+2i} = \frac{1-2i}{(1+2i)(1-2i)} = \frac{1-2i}{1+\cancel{2i}-\cancel{2i}+4\cancel{i^2}} = \frac{1}{5} - \frac{2}{5}i$$

in general, $$\frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2}$$

$z \cdot w$: multiply lengths + add angles.

z^2: Square the length + double the angle



z^3: cube the length + triple the angle...

$\sqrt{z}$: take $\sqrt{\text{length}}$ + halve the angle

$\sqrt[3]{z}$: take $\sqrt[3]{\text{length}}$ + $\frac{1}{3}$ of the angle

example: $z = 1 + 2i$ length $= \sqrt{5}$ $\theta \approx 63.4^\circ$

$$\sqrt{z} = r\cos\theta + i r \sin\theta$$

where $r = \sqrt[4]{5}$ and $\theta \approx 31.7^\circ$.