Complex numbers
\[ z = x + iy \]

```
\[
\text{multiplication:}
\begin{align*}
\text{multiply lengths} & \quad \text{and add angles} \\
\end{align*}
\]
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```
If \( z^2 = |z|^2 \)

then \( \frac{1}{z} = \frac{z}{|z|^2} \)

\[
\frac{1}{z} \text{ length is } \frac{1}{|z|} \text{ angle = opp. of what it was for z.}
\]

Euler's formula: \( e^{it} = \cos t + i \sin t \)

justify using power series:

Notice:

\[
i^2 = -1 \quad i^3 = -i \quad i^4 = 1
\]

\[
i^5 = i \quad i^6 = -1 \quad i^7 = -i \quad i^8 = 1 \text{ etc.}
\]

Recall from 253 (or wherever you learned it)

that \( e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots \)

so \( e^{it} = 1 + it + \frac{i^2 t^2}{2!} + \frac{i^3 t^3}{3!} + \frac{i^4 t^4}{4!} + \ldots \)

\[
= 1 + it - \frac{t^2}{2!} - \frac{t^3}{3!} - \frac{t^4}{4!} + \frac{t^5}{5!} - \frac{t^6}{6!} + \ldots
\]

\[
= \cos t + i \sin t
\]
Examples

\[ e^{i\pi} = \cos \pi + i \sin \pi = -1 + 0i = -1 \]

\[ e^{i\pi/2} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = 0 + i = i \]

\[ e^{i\pi/4} = \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \]

\[ w = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i \]

\[ w^3 = e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1 + 0i = 1 \]

---

Roots of unity (unity = fancy for 1)

Fix \( n \in \{1, 2, 3, \ldots\} \)

Look at all \( z \in \mathbb{C} \) with \( z^n = 1 \)

Then \( |z| = 1 \) and angle is a multiple of \( \frac{2\pi}{n} \)

Cube roots of 1:

\[ w_1 = e^{\frac{2\pi i}{3}} \]

\[ w_2 = e^{\frac{4\pi i}{3}} \]

\[ w_3 = e^{\frac{6\pi i}{3}} = e^{2\pi i} = 1 \]

Fourth roots of 1:

\[ z_1 = e^{\frac{2\pi i}{4}} \]

\[ z_2 = e^{\frac{2\pi i}{4} + \frac{2\pi i}{4}} = e^{\frac{4\pi i}{4}} = e^{\pi i} = -1 \]

\[ z_3 = e^{\frac{2\pi i}{4} + \frac{4\pi i}{4}} = e^{\frac{6\pi i}{4}} = e^{\frac{3\pi i}{2}} \]

"Primitive with root of 1" is \( e^{\frac{2\pi i}{n}} = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n} \)
Why do we care?

Just as solutions to \( x^2 = 1 \) are \( x = \pm 1 \).

Solutions to \( x^3 = 27 \) are 3 and 3w and 3w^2

where \( w = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \).

Example from worksheet last time:

Solve \( z^3 = i \):

One soli:

\[ z = \cos 30^\circ + i \sin 30^\circ = \frac{\sqrt{3}}{2} + \frac{1}{2}i \]

Two more: that \( w \) and that \( w^2 \).