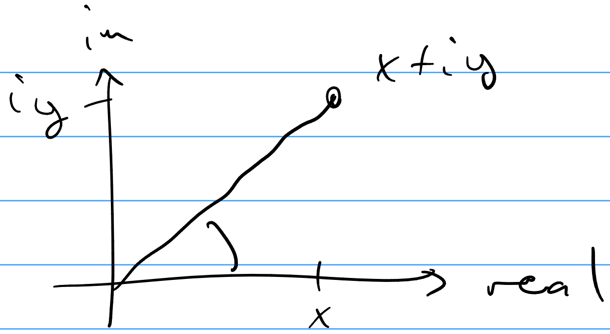
Complex numbers

$z = x + iy$



multiplication:

multiply lengths

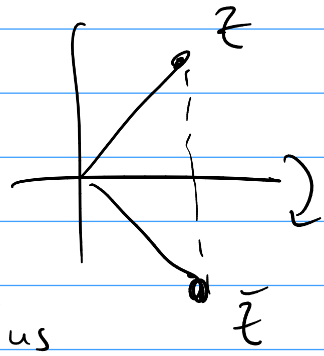
+ add angles

"Argand diagrams"

useful:

① complex conjugate $\bar{z} = x - iy$

check: $\overline{zw} = \bar{z} \cdot \bar{w}$ $\quad \overline{z+w} = \bar{z} + \bar{w}$



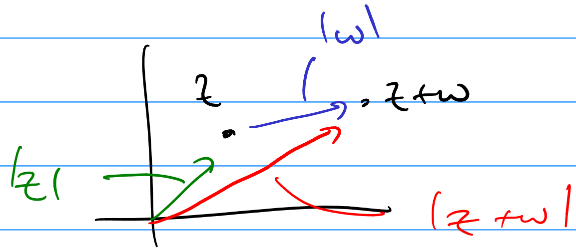
② length or abs. value or modulus

$|z| = \sqrt{x^2+y^2} \in \mathbb{R}$

check: $|zw| = |z||w|$ v. useful.

also have the triangle ineq.

$|z+w| \le |z| + |w|$

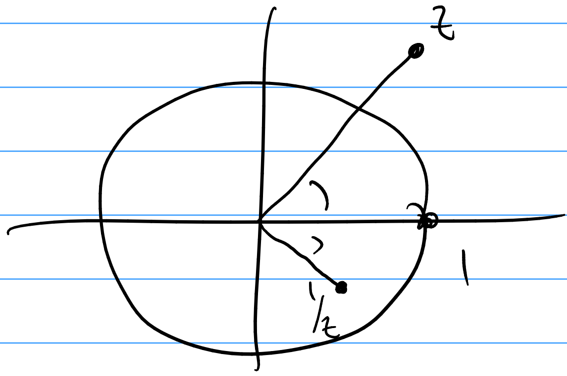


they go together: $z \cdot \bar{z} = |z|^2$

check: $(x+iy)(x-iy) = x^2 \cancel{-ixy} \cancel{+ixy} \underbrace{-i^2}_{\to +1} y^2$

if $z\bar{z} = |z|^2$

then $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$



$\frac{1}{z}$: length is $\frac{1}{|z|}$ angle = opp. of what it was for z.

Euler's formula: $e^{it} = \cos t + i \sin t$

justify using power series:

notice: $i^2 = -1$ $i^3 = -i$ $i^4 = +1$

$i^5 = i$ $i^6 = -1$ $i^7 = -i$ $i^8 = 1$ etc.

recall from 253 (or wherever you learned it)

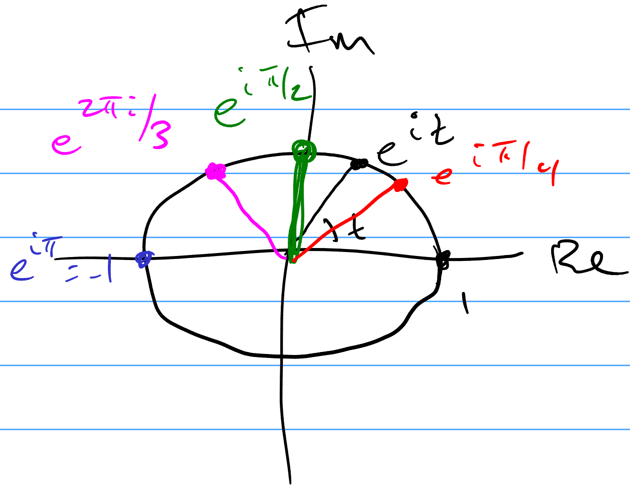
that $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$

so $e^{it} = 1 + it + \frac{i^2t^2}{2!} + \frac{i^3t^3}{3!} + \frac{i^4t^4}{4!} + \cdots$

$= 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + i\frac{t^5}{5!} - \frac{t^6}{6!} - i\frac{t^7}{7!} + \cdots$

$= \cos t + i \sin t$

examples



$e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0i = -1$

$e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = 0 + i = i$

$e^{i\pi/4} = \sqrt{2}/2 + i\sqrt{2}/2$

$\omega = e^{2\pi i/3} = -1/2 + \frac{\sqrt{3}}{2} i$

$\omega^3 = e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1 + 0i = 1$

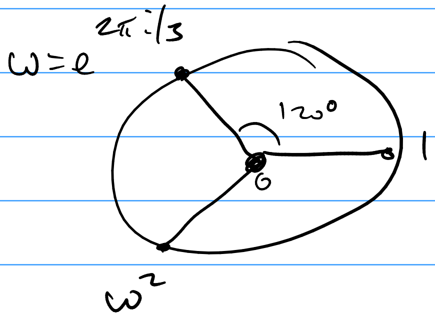
Roots of unity (unity = fancy for 1)

fix $n \in \{1, 2, 3, \ldots\}$

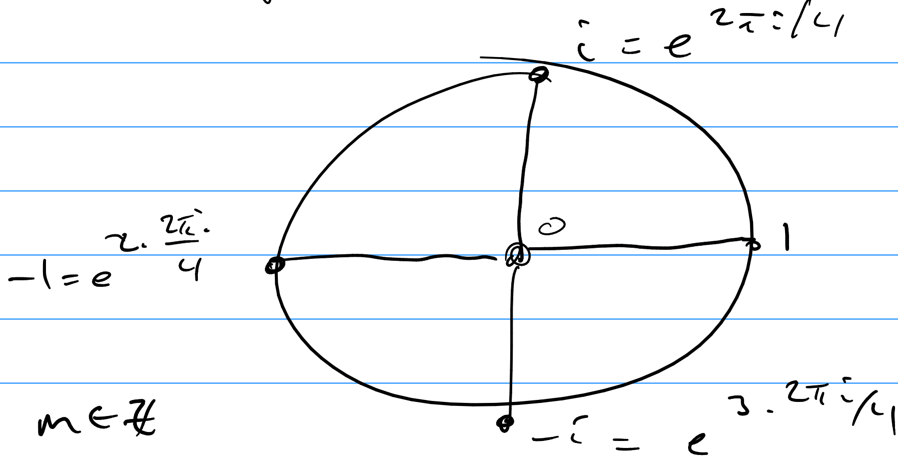
look at all $\zeta \in \mathbb{C}$ with $\zeta^n = 1$ ("zeta")

then $|\zeta| = 1$ and angle is a multiple of $\frac{2\pi}{n}$

cube roots of 1:



fourth roots of 1:



$\rightsquigarrow \zeta = e^{2\pi i \cdot m/n}$ some $m \in \mathbb{Z}$

"primitive nth root of 1" is $e^{2\pi i/n} = \cos\frac{2\pi}{n} + i\sin 2\pi/n$

Why do we care?

Just as sols to $x^2 = 4$

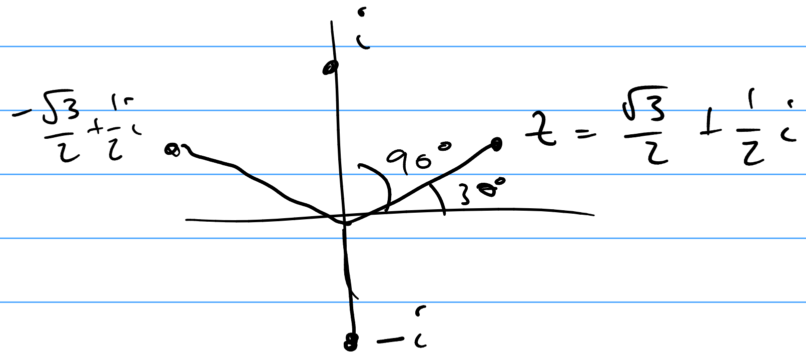
are $x = \pm 2$

sols to $x^3 = 27$ are 3 and 3ω and $3\omega^2$

where $\omega = e^{2\pi i/3} = \frac{-1}{2} + \frac{\sqrt{3}}{2} i$

Example from worksheet last time:

solve $z^3 = i$



one sol:

$z = \cos 30° + i \sin 30° = \frac{\sqrt{3}}{2} + \frac{1}{2} i$

two more: that $\cdot\, \omega$ and that $\cdot\, \omega^2$