Last time, finding roots of \( z^2 - 3z + 1 = 0 \)

\[
\text{Subbed } z = v + \frac{1}{v}
\]

\[
\Rightarrow v^3 + 1 + \frac{1}{v^3} = 0
\]

\[
\Rightarrow (v^3)^2 + v^3 + 1 = 0
\]

\[
v^3 = -\frac{1}{2} \pm \frac{\sqrt{3}i}{2}
\]

\[
v^3 = \frac{2\pi i}{3}
\]

\[
v = e^{\frac{2\pi i}{3}} = \cos 120^\circ + i \sin 120^\circ
\]

\[
v = e^{\frac{2\pi i}{3}} = \cos 40^\circ + i \sin 120^\circ
\]

\[
v = e^{\frac{2\pi i}{3}} = \cos 160^\circ + i \sin 160^\circ
\]

\[
v = e^{\frac{2\pi i}{3}} = \cos 280^\circ + i \sin 280^\circ
\]

Now \( z = v + \frac{1}{v} \), what is \( \frac{1}{v} \)?

\[
\frac{1}{v} = e^{-\frac{2\pi i}{3}} = \cos (-40^\circ) + i \sin (-40^\circ)
\]

\[
\frac{1}{v} = \cos 40^\circ - i \sin 40^\circ
\]

\[
\frac{1}{v} = \cos 160^\circ - i \sin 160^\circ
\]

\[
\frac{1}{v} = \cos 280^\circ - i \sin 280^\circ
\]

\[
z = v + \frac{1}{v} = 2 \cos 40^\circ \text{ (imaginary part cancels!)}
\]

\[
\text{or } 2 \cos 160^\circ \text{ or } 2 \cos 280^\circ
\]
Polynomial Rings.

Let \( R \) be a commutative ring, e.g., \( R = \mathbb{Z} \) or \( \mathbb{Q} \) or \( \mathbb{R} \) or \( \mathbb{C} \).

Then \( R[x] = \text{ring of polynomials w/ coeffs in } R \)

\[ \{ \text{Symbols } a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \}
\text{ where } a_i \in R \}

Add and multiply in the way you know.

Check: ring axioms are satisfied.

Example: \( R = \mathbb{F}_3 = \{ \overline{0}, \overline{1}, \overline{2} \} \)

\[ f = x^2 + x = \overline{1} x^2 + \overline{1} x + \overline{0} \in \mathbb{F}_3 [x] \]

\[ g = x + \overline{2} = \overline{1} x + \overline{2} \]

Then \( f + g = x^2 + \overline{2} x + \overline{2} \)

and \( fg = x^3 + \overline{2} x^2 + \overline{1} x^2 + \overline{2} x \)

\[ \overline{1} + \overline{2} = \overline{0} \]

\[ = x^3 + \overline{2} x \quad (\text{or } \overline{1} x^3 + \overline{0} x^2 + \overline{2} x + \overline{0}) \]
a polynomial \( f \in \mathbb{R}[x] \) determines a function

\[
\mathbb{R} \longrightarrow \mathbb{R} \\
x \longrightarrow f(x)
\]

but we want to think of \( f \) as a formal symbol and not the same as \( \sigma \) that func.

Example: \( h = x^3 + 2x \in \mathbb{Z}_3[x] \) as above

has \( h(\bar{0}) = \bar{0}^3 + 2 \cdot \bar{0} = \bar{0} \)

\( h(\bar{1}) = \bar{1}^3 + 2 \cdot \bar{1} = \bar{0} \)

\( h(\bar{2}) = \bar{2}^3 + 2 \cdot \bar{2} = \bar{2} + \bar{1} = \bar{0} \)

\( h \) and \( \sigma \) are different elt.s of \( \mathbb{Z}_3[x] \),

even though they determine the same function \( \mathbb{Z}_3 \longrightarrow \mathbb{Z}_3 \).
Consider \( x^3 + 3x^2 + 2x + 1 \in \mathbb{R}(x) \).

The coefficients are \( 1, 3, 2, 1 \in \mathbb{R} \).

The terms are \( x^3, 3x^2, 2x, 1 \).

\( x^3 \) is the leading term.

A polynomial is monic if the leading coefficient is 1.

Example: that guy above was monic, but \( 4x^2 + 4x + 1 \) is not monic.

In \( \mathbb{Q}[x] \), could divide by 4 to get \( x^2 + x + \frac{1}{4} \) which has the same roots.

If \( f \in \mathbb{R}(x) \) is not zero, write \( f = a_n x^n + \text{lower terms} \) where \( a_n \neq 0 \)

Then \( \deg f = n \).

\( \deg (f+g) \leq \max \{ \deg f, \deg g \} \)

\( (x^2 + x) + (\frac{-x^2 + 3}{x^2}) = x + 3 \) with \( \deg 2 \) and \( \deg 2 \) and \( \deg 1 \).
if \( R \) is an integral domain then

\[
\deg(fg) = \deg f + \deg g
\]

pf: write \( f = a_n x^n + \text{lower-order terms} \)

\[ g = b_m x^m + \text{lower terms} \]

where \( a_n \neq 0 \) and \( b_m \neq 0 \)

then \( fg = a_n b_m x^{m+n} + \text{lower terms} \)

and \( a_n b_m \neq 0 \) (see \( R \) is an int. dom.)

\[ \square \]

Non-example: in \( \mathbb{Z}_6[x] \),

\[(2x + 3) \cdot (3x^2 + 2x) = 6x^3 + 3x^2 + 6x \]

\[ \deg 1 \quad \deg 2 \quad \deg 1, \text{ not 3!} \]

\[ 6 \equiv 0 \mod 3 \]

In \( \mathbb{Z}_6 \), we had a fact 1 (long division):

\( \forall a, b \in \mathbb{Z}_6 \) with \( b \neq 0 \), can write

\[
a = bq + r \quad \text{where} \quad 0 \leq r < b
\]

or \( \frac{a}{b} = q + \frac{r}{b} \)
Now: if $F$ is a field, same is true in $F[x]$:

For all $f, g \in F[x]$ with $g \neq 0$,

exists $q, r$ with $f = qg + r$ and $\deg r = \deg g$

Book: example: $F = \mathbb{Q}_5$

\[
\begin{array}{c}
3x + 1 \\
2x^2 + x + 1 | x^3 + 2 \\
\end{array}
\]

\[
\begin{array}{c}
\text{?} \\
\text{?} \\
\text{x + 1} \\
\end{array}
\]

See it next time?