Last time, finding roots of $z^3 - 3z + 1 = 0$

subbed $z = v + \frac{1}{v}$

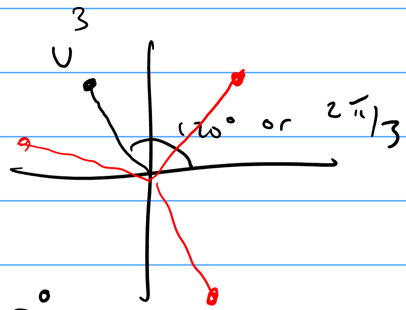
$\rightsquigarrow v^3 + 1 + \frac{1}{v^3} = 0$

$u^2 + u + 1 = 0 \ldots$ use quadr. fmla.

$\rightsquigarrow (v^3)^2 + v^3 + 1 = 0$

take + here

$$v^3 = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$



$$v^3 = e^{2\pi i/3} = \cos 120° + i \sin 120°$$

$$v = e^{2\pi i/9} = \underbrace{\cos 40° + i \sin 40°}$$

$\cdot e^{2\pi i/3}$

or $e^{2\pi i \cdot 4/9} = \cos 160° + i \sin 160°$

$\cdot e^{2\pi i/3}$

or $e^{2\pi i \cdot 7/9} = \cos 280° + i \sin 280°$

$|v| = 1$ then $\frac{1}{v} = \frac{\bar{v}}{|v|^2} = \bar{v}$

now $z = v + \frac{1}{v}$. what is $\frac{1}{v}$?

$$\frac{1}{v} = e^{-2\pi i/9} = \cos(-40°) + i \sin(-40°)$$
$$= \cos 40° - i \sin 40°$$

or $e^{2\pi i \cdot (-4/9)} = \cos 160° - i \sin 160°$

or $e^{2\pi i (-7/9)} = \cos 280° - i \sin 280°$

So $v + \frac{1}{v} = 2 \cos 40°$ (imaginary parts cancel!)

or $2 \cos 160°$ or $2 \cos 280°$

Polynomial Rings.

Let R be a commutative ring,

e.g. $R = \mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$ ...

then $R[x]$ = ring of polynomials w/ coeffs in R

$$= \left\{ \text{symbols } a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \text{ where } a_i \in R \right\}$$

add + multiply in the way you know.
check: ring axioms are satisfied.

example: $R = \mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$

$$f = x^2 + x = \bar{1}x^2 + \bar{1}x + \bar{0} \qquad \in \mathbb{Z}_3[x]$$

$$g = x + \bar{2} = \bar{1}x + \bar{2}$$

then $f + g = x^2 + \bar{2}x + \bar{2}$

and $fg = x^3 + \underbrace{\bar{2}x^2 + \bar{1}x^2}_{\bar{1}+\bar{2}=\bar{0}} + \bar{2}x$

$$= x^3 + \bar{2}x \quad (\text{or } \bar{1}x^3 + \bar{0}x^2 + \bar{2}x + \bar{0})$$

a polynomial $f \in R[x]$ determines a function

$$\begin{aligned} R &\longrightarrow R \\ x &\longmapsto f(x) \end{aligned}$$

but we want to think of f as a formal symbol and not "the same as" that func.

example: $h = x^3 + \bar{2}x \in \mathbb{Z}_3[x]$ as above

has $h(\bar{0}) = \bar{0}^3 + \bar{2} \cdot \bar{0} = \bar{0}$

$h(\bar{1}) = \bar{1}^3 + \bar{2} \cdot \bar{1} = \bar{0}$

$h(\bar{2}) = \bar{2}^3 + \bar{2} \cdot \bar{2} = \bar{2} + \bar{1} = \bar{0}$

h and 0 are different elts. of $\mathbb{Z}_3[x]$

even though they determine the same function $\mathbb{Z}_3 \longrightarrow \mathbb{Z}_3$.

Consider $x^3 + 3x^2 + 2x + 1 \in \mathbb{Z}[x]$

the leading coefficient

the coefficients are $1, 3, 2, 1 \in \mathbb{Z}$

the terms are $x^3, 3x^2, 2x, 1$

the leading term

a polynomial is monic if the leading coeff. is 1.

example: that guy above was monic, but

$4x^2 + 4x + 1$ is not monic.

in $\mathbb{Q}[x]$, could divide by 4 to get

$x^2 + x + \frac{1}{4}$ which has the same roots.

if $f \in R[x]$ is not zero,

write $f = a_n x^n$ + lower terms

where $a_n \neq 0$

then $\deg f = n$

$\deg(f+g) \leq \max\{\deg f, \deg g\}$

$$\underbrace{(x^2 + x)}_{\deg 2} + \underbrace{(-x^2 + 3)}_{\deg 2} = \underbrace{x + 3}_{\deg 1}$$

($\deg 0 = -\infty$ if you want)

if R is an integral domain then

$$\deg(fg) = \deg f + \deg g$$

pf: write $f = a_n x^n$ + lower-order terms

$g = b_m x^m$ + lower terms

where $a_n \neq 0$ and $b_m \neq 0$

then $fg = a_n b_m x^{m+n}$ + lower terms

and $a_n b_m \neq 0$ bec. R is an int. dom.

$\boxtimes$

non-example: in $\mathbb{Z}_6[x]$,

$$\underset{\deg 1}{(\bar{2}x + \bar{3})}\ \underset{\deg 2}{(\bar{3}x^2 + \bar{2}x)} = \underset{\deg 2,\ \text{not } 3!}{\bar{0}x^3 + \bar{3}x^2 + \bar{0}x}$$

bec. $\bar{2} \cdot \bar{3} = \bar{0}$

In $\mathbb{Z}$, we had a fact (long division):

$\forall a, b \in \mathbb{Z}$ with $b \neq 0$, can write

$$a = \underset{\text{quotient}}{b q} + \underset{\text{remainder}}{r} \quad \text{where } 0 \leq r < b$$

or $\frac{a}{b} = q + \frac{r}{b}$

Now: if F is a field, same is true in $F[x]$:

$\forall\, f, g \in F[x]$ with $g \neq 0$,

$\exists\, q, r$ with $f = qg + r$ and $\deg r < \deg g$

Book: example: $F = \mathbb{Z}_5$

$$\begin{array}{r l} & \bar{3}x + \bar{1} \\ \bar{2}x^2 + x + \bar{1} \,\big) & x^3 + \bar{2} \\ & \vdots \\ & x + \bar{1} \end{array}$$

See it next time?