How to tell if a polynomial is irreducible?

In $\mathbb{C}[x]$, irred. = linear

In $\mathbb{R}[x]$, irred = either linear or quadratic w/o real roots

In $\mathbb{Q}[x]$, complicated. Need some tools.

First, does it have any roots in $\mathbb{Q}$?

Rational Root Theorem

(1) let $f = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ $a_i \in \mathbb{Z}$

monic

if f has a root in $\mathbb{Q}$, the root was actually in $\mathbb{Z}$, and it divides a_0.

Example: if $f = x^3 - x^2 - 10x - 8$

possible roots are $\pm 1, \pm 2, \pm 4, \pm 8$

$f(1) = -18$ $f(-1) = 0$

$f(2) = -24$ $f(-2) = 0$

$f(4) = 0$

$f = (x+1)(x+2)(x-4)$

example: $f = x^3 - 5$ plug in $\pm 1, \pm 5$

$\rightsquigarrow$ never get 0

so $\sqrt[3]{5}$ must be irrational

(2) more generally, if

$f = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$

has a root $\frac{r}{s} \in \mathbb{Q}$ $\quad \gcd(r,s) = 1$

then $r \mid a_0$ and $s \mid a_n$

Example: $f = 15x^4 + 4x^3 + 11x^2 + 4x - 4$,

try $\pm 1, \pm 2, \pm 4, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}$

$\pm \frac{1}{5}, \pm \frac{2}{5}, \pm \frac{4}{5}, \pm \frac{1}{15}, \pm \frac{2}{15}, \pm \frac{4}{15}$

do it $\rightsquigarrow$ $-2/3$ and $2/5$ work

$f = (x + 2/3)(x - 2/5)(\text{irred. quadratic})$

Proof: suppose that $\frac{r}{s}$ is a root of $a_n x^n + \cdots + a_0$

plug it in: $a_n \frac{r^n}{s^n} + a_{n-1} \frac{r^{n-1}}{s^{n-1}} + \cdots + a_1 \frac{r}{s} + a_0 = 0$ $\quad \cdot s^n$

clear denominators: $a_n r^n + a_{n-1} r^{n-1} s + a_{n-2} r^{n-2} s^2 + \cdots + a_1 r s^{n-1} + a_0 s^n = 0$

So $\underline{a_0 s^n}$ is a multiple of $\underline{r}$ $\quad (r \mid a_0 s^n)$

and $a_n r^n$ is a multiple of s $\quad (s \mid a_n r^n)$

bec. $\gcd(r,s) = 1$, have $r \mid a_0$ and $s \mid a_n$. $\square$

2^{nd} tool: Gauss's lemma:

$f \in \mathbb{Z}[x]$ factors in $\mathbb{Q}[x]$ iff it factors in $\mathbb{Z}[x]$

Example: $f = 15x^4 + 4x^3 + 11x^2 + 4x - 4$

factors in $\mathbb{Q}[x]$ as $(x+2/3)(x-2/5)(15x^2+15)$

but could have factored it in $\mathbb{Z}[x]$ as

$(3x+2)(5x-2)(x^2+1)$.