Solutions to Homework 1

§1.1 #4. You only had to do four parts, but I'll do them all.

a. *The sum of the first n positive integers is* $\frac{n(n+1)}{2}$.

For $n = 1$ the claim is that $1 = \frac{1(1+1)}{2}$, which is true. Suppose the claim holds for n; then

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{(n+2)(n+1)}{2}, \end{aligned}$$

so the claim holds for $n + 1$.

b. *The sum of the first n odd integers is* n^2.

For $n = 1$ the claim is that $1 = 1^2$, which is true. Suppose the claim holds for n; then

$$\begin{aligned} 1 + 3 + \cdots + (2n-1) + (2n+1) &= n^2 + (2n+1) \\ &= (n+1)^2, \end{aligned}$$

so the claim holds for $n + 1$.

c. *The sum of the squares of the first n positive integers is* $\frac{n(n+1)(2n+1)}{6}$.

For $n = 1$ the claim is that $1^2 = \frac{1(1+1)(2\cdot 1+1)}{6}$, which is true. Suppose the claim holds for n; then

$$\begin{aligned}
1^2 + 2^2 + \cdots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= \frac{2n^3 + 3n^2 + n}{6} + \frac{6n^2 + 12n + 6}{6} \\
&= \frac{2n^3 + 9n^2 + 13n + 6}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6},
\end{aligned}$$

so the claim holds for $n + 1$.

d. *For $n \geq 1$, $n^3 - n$ is divisible by 3.*

For $n = 1$ the claim is that $1^3 - 1 = 0$ is divisible by 3, which is true. Suppose the claim holds for n. We have

$$\begin{aligned}
(n+1)^3 - (n+1) &= n^3 + 3n^2 + 3n + 1 - n - 1 \\
&= \underline{n^3 - n} + \underline{3n^2 + 3n}.
\end{aligned}$$

Of the underlined terms, the first is divisible by 3 by our inductive hypothesis, and the second is clearly divisible by 3; thus the claim holds for $n + 1$.

(A better proof, not by induction: factor $n^3 - n$ as $n(n+1)(n-1)$, and observe that any n is either a multiple of 3, or 1 more than a multiple of 3, or 1 less than a multiple of 3; thus one of the three factors is a multiple of 3.)

e. *For $n \geq 3$, $n + 4 < 2^n$.*

For $n = 3$ the claim is that $3 + 4 < 2^3$, i.e. that $7 < 8$, which is true. Suppose the claim holds for n. We have

$$2^{n+1} = 2 \cdot 2^n > 2(n+4) = 2n + 8 \geq n + 5,$$

where in the last inequality we have used the fact that $n \geq 0$ and $8 \geq 5$. Thus the claim holds for $n + 1$.

f. *For all $n \in \mathbb{N}$, $1 + 5 + 9 + \cdots + (4n + 1) = (2n + 1)(n + 1)$.*

For $n = 1$ the claim is that $1 + 5 = 3 \cdot 2$, which is true. Suppose the claim holds for n; then

$$\begin{aligned} 1 + 5 + \cdots + (4n + 1) + (4n + 5) &= (2n + 1)(4n + 1) + (4n + 5) \\ &= 2n^2 + 7n + 6 \\ &= (2n + 3)(n + 2), \end{aligned}$$

so the claim holds for $n + 1$.

g. *For any positive integer n, one of n, $n+2$, and $n+4$ must be divisible by 3.*

For $n = 1$ the claim is that one of $1, 3, 5$ is divisible by 3, which is true. Suppose the claim holds for n. If n is divisble by 3 then $n + 3 = (n + 1) + 2$ is divisible by 3. If $n + 2$ is divisible by 3 then $(n+2)+3 = (n+1)+4$ is divisible by 3. If $n+4$ is divisible by 3 then $(n + 4) - 3 = n + 1$ is divisible by 3. Thus the claim holds for $n + 1$.

h. *$3^n \geq 2n + 1$ for all $n \in \mathbb{N}$.*

For $n = 1$ the claim is that $3 \geq 3$, which is true. Suppose the claim holds for n; then

$$3^{n+1} = 3 \cdot 3^n \geq 3(2n + 1) = 6n + 3 \geq 2n + 3,$$

so the claim holds for $n + 1$.

i. *For all $n \in \mathbb{N}$, $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n}$.*

For $n = 1$ the claim is that $\frac{1}{2} = 1 - \frac{1}{2}$, which is true. Suppose the claim holds for n; then

$$\begin{aligned} &\tfrac{1}{n+2} + \tfrac{1}{n+3} + \cdots + \tfrac{1}{2n} + \tfrac{1}{2n+1} + \tfrac{1}{2n+2} \\ &= \left(\tfrac{1}{n+1} + \tfrac{1}{n+2} + \tfrac{1}{n+3} + \cdots + \tfrac{1}{2n}\right) - \tfrac{1}{n+1} + \tfrac{1}{2n+1} + \tfrac{1}{2n+2} \\ &= \left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \cdots + \tfrac{1}{2n-1} - \tfrac{1}{2n}\right) - \tfrac{2}{2n+2} + \tfrac{1}{2n+1} + \tfrac{1}{2n+2} \\ &= \left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \cdots + \tfrac{1}{2n-1} - \tfrac{1}{2n}\right) + \tfrac{1}{2n+1} - \tfrac{1}{2n+2}, \end{aligned}$$

so the claim holds for $n + 1$.

j. *Prove that for any $x > -1$ and any $n \in \mathbb{N}$, $(1 + x)^n \geq 1 + nx$.*

For $n = 1$ the claim is that for all $x > -1$ we have $1+x \geq 1+x$, which is true. Suppose the claim holds for n. We have

$$\begin{aligned}(1 + x)^{n+1} = (1 + x)^n(1 + x) \geq (1 + nx)(1 + x) \\ = 1 + (n + 1)x + nx^2 \geq 1 + (n + 1)x,\end{aligned}$$

where in the first inequality we have used the inductive hypothesis $(1 + x)^n \geq (1 + nx)$ and the fact that $1 + x > 0$, and in the second inequality we have used the fact that $n \geq 0$ and $x^2 \geq 0$. Thus the claim holds for $n + 1$.

§1.1 #5. *Prove that for every $n \in \mathbb{N}$, $\sum_{j=0}^{n} \binom{n}{j} = 2^n$.*
By definition, $\binom{n}{j}$ is the number of j-element subsets of the set $\{1, 2, \ldots, n\}$. Thus $\sum_{j=0}^{n} \binom{n}{j}$ is the number of subsets $\{1, 2, \ldots, n\}$ of any size whatsoever. Such a subset is described by going through the list $1, 2, \ldots, n$ and saying "yes" or "no" to each one: either 1 is in the subset or it's not; either 2 is in the subset or it's not; and so on. This gives

$$\overbrace{2 \cdot 2 \cdot \cdots \cdot 2}^{n \text{ times}} = 2^n,$$

possible choices.

Another possible proof: rather than using the definition of $\binom{n}{j}$, we could have used the binomial theorem (Theorem 1.4). Taking $x = y = 1$, it says

$$(1 + 1)^n = \sum_{j=0}^{n} \binom{n}{j} 1^{n-j} 1^j.$$

The left-hand side is 2^n, and the right-hand side is the sum we want, because $1^{n-j} = 1^j = 1$.

§1.1 #5. *The Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... is obtained by the following recursive formula:* $a_1 = 1$, $a_2 = 1$, *and* $a_{n+1} = a_n + a_{n-1}$. *Prove by induction that*

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

For brevity let

$$\varphi = \frac{1+\sqrt{5}}{2} \qquad\qquad \bar{\varphi} = \frac{1-\sqrt{5}}{2}.$$

The key observation is that

$$\varphi^2 = \varphi + 1 \qquad \text{and} \qquad \bar{\varphi}^2 = \bar{\varphi} + 1.$$

The claim is that $a_n = \frac{1}{\sqrt{5}}(\varphi^n - \bar{\varphi}^n)$. We have $\varphi - \bar{\varphi} = \sqrt{5}$, so the claim holds for $n = 1$, and

$$\varphi^2 - \bar{\varphi}^2 = (\varphi + 1) - (\bar{\varphi} + 1) = \varphi - \bar{\varphi} = \sqrt{5},$$

so the claim holds for $n = 2$. Now suppose the claim holds for n and $n - 1$; then

$$\begin{aligned}
a_{n+1} &= a_n + a_{n-1} \\
&= \tfrac{1}{\sqrt{5}}(\varphi^n - \bar{\varphi}^n) + \tfrac{1}{\sqrt{5}}(\varphi^{n-1} - \bar{\varphi}^{n-1}) \\
&= \tfrac{1}{\sqrt{5}}(\varphi^n + \varphi^{n-1} - \bar{\varphi}^n - \bar{\varphi}^{n-1}) \\
&= \tfrac{1}{\sqrt{5}}(\varphi^{n-1}(\varphi + 1) - \bar{\varphi}^{n-1}(\bar{\varphi} + 1)) \\
&= \tfrac{1}{\sqrt{5}}(\varphi^{n-1} \cdot \varphi^2 - \bar{\varphi}^{n-1} \cdot \bar{\varphi}^2) \\
&= \tfrac{1}{\sqrt{5}}(\varphi^{n+1} - \bar{\varphi}^{n+1}),
\end{aligned}$$

so the claim holds for $n + 1$.