§1.1 #4. You only had to do four parts, but I’ll do them all.

a. *The sum of the first n positive integers is* $\frac{n(n+1)}{2}$.

For $n = 1$ the claim is that $1 = \frac{1(1+1)}{2}$, which is true. Suppose the claim holds for $n$; then

$$1 + 2 + \cdots + n + (n + 1) = \frac{n(n + 1)}{2} + (n + 1)$$

$$= \frac{n(n + 1)}{2} + \frac{2(n + 1)}{2}$$

$$= \frac{(n + 2)(n + 1)}{2},$$

so the claim holds for $n + 1$.

b. *The sum of the first n odd integers is* $n^2$.

For $n = 1$ the claim is that $1 = 1^2$, which is true. Suppose the claim holds for $n$; then

$$1 + 3 + \cdots + (2n - 1) + (2n + 1) = n^2 + (2n + 1)$$

$$= (n + 1)^2,$$

so the claim holds for $n + 1$. 
c. The sum of the squares of the first $n$ positive integers is $\frac{n(n+1)(2n+1)}{6}$.

For $n = 1$ the claim is that $1^2 = \frac{1(1+1)(2\cdot1+1)}{6}$, which is true. Suppose the claim holds for $n$; then

$$1^2 + 2^2 + \cdots + n^2 + (n + 1)^2 = \frac{n(n + 1)(2n + 1)}{6} + (n + 1)^2$$

$$= \frac{2n^3 + 3n^2 + n}{6} + \frac{6n^2 + 12n + 6}{6}$$

$$= \frac{2n^3 + 9n^2 + 13n + 6}{6}$$

$$= \frac{(n + 1)(n + 2)(2n + 3)}{6},$$

so the claim holds for $n + 1$.

d. For $n \geq 1$, $n^3 - n$ is divisible by 3.

For $n = 1$ the claim is that $1^3 - 1 = 0$ is divisible by 3, which is true. Suppose the claim holds for $n$. We have

$$(n + 1)^3 - (n + 1) = n^3 + 3n^2 + 3n + 1 - n - 1$$

$$= n^3 - n + 3n^2 + 3n.$$

Of the underlined terms, the first is divisible by 3 by our inductive hypothesis, and the second is clearly divisible by 3; thus the claim holds for $n + 1$.

(A better proof, not by induction: factor $n^3 - n$ as $n(n+1)(n-1)$, and observe that any $n$ is either a multiple of 3, or 1 more than a multiple of 3, or 1 less than a multiple of 3; thus one of the three factors is a multiple of 3.)

e. For $n \geq 3$, $n + 4 < 2^n$.

For $n = 3$ the claim is that $3 + 4 < 2^3$, i.e. that $7 < 8$, which is true. Suppose the claim holds for $n$. We have

$$2^{n+1} = 2 \cdot 2^n > 2(n + 4) = 2n + 8 \geq n + 5,$$

where in the last inequality we have used the fact that $n \geq 0$ and $8 \geq 5$. Thus the claim holds for $n + 1$. 

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f. For all $n \in \mathbb{N}$, $1 + 5 + 9 + \cdots + (4n + 1) = (2n + 1)(n + 1)$.

For $n = 1$ the claim is that $1 + 5 = 3 \cdot 2$, which is true. Suppose the claim holds for $n$; then

$$1 + 5 + \cdots + (4n + 1) + (4n + 5) = (2n + 1)(4n + 1) + (4n + 5)$$
$$= 2n^2 + 7n + 6$$
$$= (2n + 3)(n + 2),$$

so the claim holds for $n + 1$.

g. For any positive integer $n$, one of $n$, $n + 2$, and $n + 4$ must be divisible by 3.

For $n = 1$ the claim is that one of 1, 3, 5 is divisible by 3, which is true. Suppose the claim holds for $n$. If $n$ is divisible by 3 then $n + 3 = (n + 1) + 2$ is divisible by 3. If $n + 2$ is divisible by 3 then $(n + 2) + 3 = (n + 1) + 4$ is divisible by 3. If $n + 4$ is divisible by 3 then $(n + 4) - 3 = n + 1$ is divisible by 3. Thus the claim holds for $n + 1$.

h. $3^n \geq 2n + 1$ for all $n \in \mathbb{N}$.

For $n = 1$ the claim is that $3 \geq 3$, which is true. Suppose the claim holds for $n$; then

$$3^{n+1} = 3 \cdot 3^n \geq 3(2n + 1) = 6n + 3 \geq 2n + 3,$$

so the claim holds for $n + 1$.

i. For all $n \in \mathbb{N}$, $\frac{1}{n} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n}$.

For $n = 1$ the claim is that $\frac{1}{2} = 1 - \frac{1}{2}$, which is true. Suppose the claim holds for $n$; then

$$\frac{1}{n+1} + \frac{1}{n+3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$$
$$= \left( \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} \right) - \frac{1}{n+1} + \frac{1}{2n+1} + \frac{1}{2n+2}$$
$$= \left( 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n} \right) - \frac{2}{2n+2} + \frac{1}{2n+1} + \frac{1}{2n+2}$$
$$= \left( 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n} \right) + \frac{1}{2n+1} - \frac{1}{2n+2},$$

so the claim holds for $n + 1$. 

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j. Prove that for any \( x > -1 \) and any \( n \in \mathbb{N} \), \( (1 + x)^n \geq 1 + nx \).

For \( n = 1 \) the claim is that for all \( x > -1 \) we have \( 1 + x \geq 1 + x \), which is true. Suppose the claim holds for \( n \). We have

\[
(1 + x)^{n+1} = (1 + x)^n(1 + x) \geq (1 + nx)(1 + x) = 1 + (n + 1)x + nx^2 \geq 1 + (n + 1)x,
\]

where in the first inequality we have used the inductive hypothesis \( (1 + x)^n \geq (1 + nx) \) and the fact that \( 1 + x > 0 \), and in the second inequality we have used the fact that \( n \geq 0 \) and \( x^2 \geq 0 \). Thus the claim holds for \( n + 1 \).

§1.1 #5. Prove that for every \( n \in \mathbb{N} \), \( \sum_{j=0}^{n} \binom{n}{j} = 2^n \).

By definition, \( \binom{n}{j} \) is the number of \( j \)-element subsets of the set \( \{1, 2, \ldots, n\} \). Thus \( \sum_{j=0}^{n} \binom{n}{j} \) is the number of subsets \( \{1, 2, \ldots, n\} \) of any size whatsoever. Such a subset is described by going through the list \( 1, 2, \ldots, n \) and saying “yes” or “no” to each one: either 1 is in the subset or it’s not; either 2 is in the subset or it’s not; and so on. This gives

\[
\underbrace{2 \cdot 2 \cdot \ldots \cdot 2}_{\text{n times}} = 2^n,
\]

possible choices.

Another possible proof: rather than using the definition of \( \binom{n}{j} \), we could have used the binomial theorem (Theorem 1.4). Taking \( x = y = 1 \), it says

\[
(1 + 1)^n = \sum_{j=0}^{n} \binom{n}{j} 1^{n-j} 1^j.
\]

The left-hand side is \( 2^n \), and the right-hand side is the sum we want, because \( 1^{n-j} = 1^j = 1 \).
§1.1 #5. The Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... is obtained by the following recursive formula: \( a_1 = 1, a_2 = 1, \) and \( a_{n+1} = a_n + a_{n-1}. \) Prove by induction that

\[
a_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right).
\]

For brevity let

\[
\varphi = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \bar{\varphi} = \frac{1 - \sqrt{5}}{2}.
\]

The key observation is that

\[\varphi^2 = \varphi + 1 \quad \text{and} \quad \bar{\varphi}^2 = \bar{\varphi} + 1.\]

The claim is that \( a_n = \frac{1}{\sqrt{5}} (\varphi^n - \bar{\varphi}^n). \) We have \( \varphi - \bar{\varphi} = \sqrt{5}, \) so the claim holds for \( n = 1, \) and

\[\varphi^2 - \bar{\varphi}^2 = (\varphi + 1) - (\bar{\varphi} + 1) = \varphi - \bar{\varphi} = \sqrt{5},\]

so the claim holds for \( n = 2. \) Now suppose the claim holds for \( n \) and \( n - 1; \) then

\[
a_{n+1} = a_n + a_{n-1}
= \frac{1}{\sqrt{5}} (\varphi^n - \bar{\varphi}^n) + \frac{1}{\sqrt{5}} (\varphi^{n-1} - \bar{\varphi}^{n-1})
= \frac{1}{\sqrt{5}} (\varphi^n + \varphi^{n-1} - \bar{\varphi}^n - \bar{\varphi}^{n-1})
= \frac{1}{\sqrt{5}} (\varphi^{n-1}(\varphi + 1) - \varphi^{n-1}(\bar{\varphi} + 1))
= \frac{1}{\sqrt{5}} (\varphi^{n-1} \cdot \varphi^2 - \varphi^{n-1} \cdot \bar{\varphi}^2)
= \frac{1}{\sqrt{5}} (\varphi^{n+1} - \bar{\varphi}^{n+1}),
\]

so the claim holds for \( n + 1. \)