Worksheet 10

Math 391, Abstract Algebra

Wednesday, October 21, 2020

This is based on Shifrin §1.4 #6 and #10.

Here are the ring axioms, copied from page 38:

(1) Addition is commutative: for all $a, b \in R$ we have $a + b = b + a$.

(2) Addition is associative: for all $a, b, c \in R$ we have $(a+b)+c = a+(b+c)$.

(3) Additive identity: There is an element $0 \in R$ such that for all $a \in R$ we have $0 + a = a$.

(4) Additive inverses: For each $a \in R$ there is an element $-a \in R$ such that $a + (-a) = 0$.

(5) Multiplication is associative: for all $a, b, c \in R$ we have $(ab)c = a(bc)$.

(6) Multiplicative unit: There is an element $1 \in R$, different from 0, such that for all $a \in R$ we have $a \cdot 1 = 1 \cdot a = a$.

(7) Distributivity: For all $a, b, c \in R$ we have $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.

Show that the familiar facts below follow from the ring axioms. With your colleagues, decide who will work on each one, and then present your answers to each other. Or if something gives you trouble, discuss it together.

(a) The additive identity is unique: If $0' \in R$ is another element such that for all $a \in R$ we have $0' + a = a$, then $0' = 0$.

Hint: Consider $0 + 0'$.

(b) The multiplicative identity is unique: If $1' \in R$ is another element such that for all $a \in R$ we have $a \cdot 1' = 1' \cdot a = a$, then $1' = 1$.

Hint: Consider $1 \cdot 1'$.

(c) Additive inverses are unique: given some $a \in R$, if $b, c \in R$ satisfy $a + b = 0$ and $a + c = 0$, then $b = c$.

Hint: Consider $b + a + c$.

(d) Multiplicative inverses are unique, when they exist: given some $a \in R$, if $b, c \in R$ satisfy $a \cdot b = b \cdot a = 1$ and $a \cdot c = c \cdot a = 1$, then $b = c$.

Hint: Consider $b \cdot a \cdot c$.

(e) For all $a \in R$ we have $0 \cdot a = a \cdot 0 = 0$.

Hint: Simplify $(0 + 0) \cdot a$ in two ways. Add $-(0 \cdot a)$ to both sides.

(f) For all $a \in R$ we have $-(-a) = a$.

(g) For all $a \in R$ we have $-a = (-1) \cdot a = a \cdot (-1)$.

Hint: Simplify $(1 + -1) \cdot a$ in two different ways and use part (d).

(h) For all $a, b \in R$ we have $-(a + b) = -a + -b$.

(i) For all $a, b \in R$ we have $(-a) \cdot b = a \cdot (-b) = -(a \cdot b)$.

(j) For all $a, b \in R$ we have $(-a) \cdot (-b) = a \cdot b$.