# Worksheet 12

Math 391, Abstract Algebra

Wednesday, October 28, 2020

We haven't studied polynomials in earnest yet, but you know what they are: we let $\mathbb{R}[x]$ denote the ring of polynomials with coefficients in $\mathbb{R}$, which is the set of symbols

$$p(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0,$$

where the coefficients $a_i$ are real numbers, and addition and multiplication work in the way you expect.

Just as we built the field $\mathbb{Q}$ from the ring $\mathbb{Z}$, we can build the field of rational functions $\mathbb{R}(x)$ from the ring of polynomials $\mathbb{R}[x]$. Notice the distinction between round brackets and square brackets. A typical element of $\mathbb{R}(x)$ is a quotient of two polynomials

$$\frac{p(x)}{q(x)} = \frac{a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0}{b_m x^m + \cdots + b_2 x^2 + b_1 x + b_0},$$

where the denominator is not zero, and addition and multiplication work in the way you expect.

I claim that $\mathbb{R}(x)$ can be given the structure of an ordered field by declaring that $\frac{p(x)}{q(x)} > 0$ if and only if the leading coefficients satisfy $\frac{a_n}{b_m} > 0$.

1. With the ordering just described, all the following rational functions are positive:

$$\frac{1}{1} \quad \frac{2}{1} \quad \frac{3}{1} \quad \frac{5x-1}{1} \quad \frac{1}{x} \quad \frac{1}{2x+1} \quad \frac{2x}{x-1} \qquad \frac{2x}{x+1} \quad \frac{4x^2}{x+3}.$$

   Put them in order from least to greatest. (By definition, $a < b$ means that $b - a > 0$.)

2. Show that this ordering is compatible with multiplication: if $\frac{p(x)}{q(x)} > 0$ and $\frac{f(x)}{g(x)} > 0$, then $\frac{p(x)}{q(x)} \cdot \frac{f(x)}{g(x)} > 0$.

3. Show that this ordering is compatible with addition: if $\frac{p(x)}{q(x)} > 0$ and $\frac{f(x)}{g(x)} > 0$, then $\frac{p(x)}{q(x)} + \frac{f(x)}{g(x)} > 0$.

4. Show that this ordering is well-defined: if $\frac{p(x)}{q(x)} = \frac{f(x)}{g(x)}$ and $\frac{p(x)}{q(x)} > 0$, then $\frac{f(x)}{g(x)} > 0$. (Really we should have done this first.)

5. Challenge: We could have put a different ordering on $\mathbb{R}(x)$, by declaring $\frac{p(x)}{q(x)} > 0$ in the new ordering if and only if $\frac{p(1/x)}{q(1/x)} > 0$ in the old ordering. Redo problem 1 with this new ordering. It's not just the reverse of what you had before!