Solutions to Worksheet 12

Math 391, Abstract Algebra

We haven't studied polynomials in earnest yet, but you know what they are: we let $\mathbb{R}[x]$ denote the ring of polynomials with coefficients in $\mathbb{R}$, which is the set of symbols

$$p(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0,$$

where the coefficients a_i are real numbers, and addition and multiplication work in the way you expect.

Just as we built the field $\mathbb{Q}$ from the ring $\mathbb{Z}$, we can build the field of rational functions $\mathbb{R}(x)$ from the ring of polynomials $\mathbb{R}[x]$. Notice the distinction between round brackets and square brackets. A typical element of $\mathbb{R}(x)$ is a quotient of two polynomials

$$\frac{p(x)}{q(x)} = \frac{a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0}{b_m x^m + \cdots + b_2 x^2 + b_1 x + b_0},$$

where the denominator is not zero, and addition and multiplication work in the way you expect.

I claim that $\mathbb{R}(x)$ can be given the structure of an ordered field by declaring that $\frac{p(x)}{q(x)} > 0$ if and only if the leading coefficients satisfy $\frac{a_n}{b_m} > 0$.

1. With the ordering just described, all the following rational functions are positive:

$$\frac{1}{1} \quad \frac{2}{1} \quad \frac{3}{1} \quad \frac{5x-1}{1} \quad \frac{1}{x} \quad \frac{1}{2x+1} \quad \frac{2x}{x-1} \quad \frac{2x}{x+1} \quad \frac{4x^2}{x+3}.$$

   Put them in order from least to greatest. (By definition, $a < b$ means that $b - a > 0$.)

Solution:

$$\frac{1}{2x+1} \quad \frac{1}{x} \quad \frac{1}{1} \quad \frac{2x}{x+1} \quad \frac{2}{1} \quad \frac{2x}{x-1} \quad \frac{3}{1} \quad \frac{5x-1}{1} \quad \frac{4x^2}{x+3}.$$

As I pointed out in class, these comparisons become easier if you notice that my definition of $\frac{p(x)}{q(x)} > 0$ is equivalent to $\lim_{x\to\infty} \frac{p(x)}{q(x)} > 0$.

2. Show that this ordering is compatible with multiplication: if $\frac{p(x)}{q(x)} > 0$ and $\frac{f(x)}{g(x)} > 0$, then $\frac{p(x)}{q(x)} \cdot \frac{f(x)}{g(x)} > 0$.

   Solution: Write

$$\begin{aligned} p(x) &= ax^m + \text{lower-order terms} \\ q(x) &= bx^n + \text{lower-order terms} \\ f(x) &= cx^k + \text{lower-order terms} \\ g(x) &= dx^l + \text{lower-order terms.} \end{aligned} \tag{$*$}$$

   If $\frac{p(x)}{q(x)} > 0$ and $\frac{f(x)}{g(x)} > 0$, then by definition we have $\frac{a}{b} > 0$ and $\frac{c}{d} > 0$. Now

$$\frac{p(x)}{q(x)} \cdot \frac{f(x)}{g(x)} = \frac{p(x)f(x)}{q(x)g(x)} = \frac{ac\, x^{m+k} + \text{lower-order terms}}{bd\, x^{n+l} + \text{lower-order terms}},$$

   and we have

$$\frac{ac}{bd} = \frac{a}{b} \cdot \frac{c}{d} > 0,$$

   so $\frac{p(x)}{q(x)} \cdot \frac{f(x)}{g(x)} > 0$.

3. Show that this ordering is compatible with addition: if $\frac{p(x)}{q(x)} > 0$ and $\frac{f(x)}{g(x)} > 0$, then $\frac{p(x)}{q(x)} + \frac{f(x)}{g(x)} > 0$.

   Solution: Continue to write the leading coefficients as we did in $(*)$ above. We have

$$\frac{p(x)}{q(x)} + \frac{f(x)}{g(x)} = \frac{p(x)g(x) + q(x)f(x)}{q(x)g(x)}.$$

   We must show that the leading coefficient of the numerator of the right-hand side, divided by the leading coefficient of the denominator, is positive, assuming that $\frac{a}{b} > 0$ and $\frac{c}{d} > 0$. The leading coefficient of the denominator is bd. For the leading coefficient of the numerator, consider the degree of $p(x)g(x)$, which is $m + l$, and the degree of $q(x)f(x)$, which is $n+k$. There are three cases, depending on whether the first is greater than, less than, or equal to the second:

- If $m+l > n+k$, then the leading coefficient of the numerator is the leading coefficient of $p(x)g(x)$, which is ad. We have $\frac{ad}{bd} = \frac{a}{b} > 0$.
- If $m+l < n+k$, then the leading coefficient of the numerator is the leading coefficient of $q(x)f(x)$ which is bc. We have $\frac{bc}{bd} = \frac{c}{d} > 0$.
- If $m+l = n+k$, then the leading coefficient of the numerator is $ad+bc$. To see that these cannot cancel, notice that $\frac{a}{b} > 0$ implies that a has the same sign as b, and $\frac{c}{d} > 0$ implies that c has the same sign as d, so ad has the same sign as bc. Now we have
$$\frac{ad+bc}{bd} = \frac{a}{b} + \frac{c}{d} > 0.$$

4. Show that this ordering is well-defined: if $\frac{p(x)}{q(x)} = \frac{f(x)}{g(x)}$ and $\frac{p(x)}{q(x)} > 0$, then $\frac{f(x)}{g(x)} > 0$. (Really we should have done this first.)

   Solution: By definition, $\frac{p(x)}{q(x)} = \frac{f(x)}{g(x)}$ means that $p(x)g(x) = f(x)q(x)$. Continue to write the leading coefficients as we did in $(*)$ above. The leading coefficient of $p(x)g(x)$ is ad, and the leading coefficient of $q(x)f(x)$ is bc, so we have $ad = bc$, so $\frac{a}{b} = \frac{c}{d}$.

   Thus if $\frac{p(x)}{q(x)} > 0$ then $\frac{a}{b} > 0$, so $\frac{c}{d} > 0$, so $\frac{f(x)}{g(x)} > 0$.

5. Challenge: We could have put a different ordering on $\mathbb{R}(x)$, by declaring $\frac{p(x)}{q(x)} > 0$ in the new ordering if and only if $\frac{p(1/x)}{q(1/x)} > 0$ in the old ordering. Redo problem 1 with this new ordering. It's not just the reverse of what you had before!

   Solution:
$$\frac{5x-1}{1} \quad \frac{2x}{x-1} \quad \frac{4x^2}{x+3} \quad \frac{2x}{x+1} \quad \frac{1}{2x+1} \quad \frac{1}{1} \quad \frac{2}{1} \quad \frac{3}{1} \quad \frac{1}{x}.$$

   You might notice that in this new ordering, $\frac{p(x)}{q(x)} > 0$ is equivalent to $\lim_{x \to 0^+} \frac{p(x)}{q(x)} > 0$.