Worksheet 19

Math 391, Abstract Algebra

Monday, November 16, 2020

In $\mathbb{Q}[x]$ and $\mathbb{Z}_p[x]$, irreducible polynomials can have very high degree, but in $\mathbb{R}[x]$ the picture is much simpler: irreducible polynomials are either linear, or quadratic with no real roots. Discuss the following outline of a proof, and convince yourselves that it's correct.

1. Let
$$f = c_n x^n + \cdots + c_2 x^2 + c_1 x + c_0$$
be a non-constant polynomial with real coefficients, and let $z = a + bi$ be a complex number. Recall that the complex conjugate $\bar{z}$ is defined to be $a - bi$.

   Prove that $f(\bar{z}) = \overline{f(z)}$. (Hint: You'll want to know that $\overline{z+w} = \bar{z} + \bar{w}$ and $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$.)

   In particular, if z is a root of f then $\bar{z}$ is also a root of f.

2. If $f \in \mathbb{R}[x]$ is not constant, then it has a root $a + bi \in \mathbb{C}$ by the fundamental theorem of algebra.

   If $b = 0$ then $x - a$ divides f.

   If $b \neq 0$ then $x - (a + bi)$ divides f, and $x - (a - bi)$ divides f, so
$$(x - (a + bi))(x - (a - bi)) = x^2 - 2a + a^2 + b^2$$
divides f.

3. So irreducible polynomials in $\mathbb{R}[x]$ are either linear, or quadratic with no real roots. (Is this clear from #2, or does it need some explanation?)