Worksheet 22

Math 391, Abstract Algebra

Wednesdsay, November 25, 2020

Let R and S be rings. As a set, the Cartesian product $R \times S$ consists of pairs (r, s) where $r \in R$ and $s \in S$. Put a ring structure on $R \times S$ by defining

$$(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$$
$$(r_1, s_1) \cdot (r_2, s_2) = (r_1 \cdot r_2, s_1 \cdot s_2).$$

You can check, although it's kind of tedious, that this definition of $+$ and $\cdot$ satisfies the ring axioms.

1. What is the additive identity in $R \times S$?

   What is the multiplicative identity?

2. Let $\varphi \colon \mathbb{Z}_6 \to \mathbb{Z}_2 \times \mathbb{Z}_3$ be the map given by

   $$\varphi(\bar{a}) = (\bar{a},\, \bar{a}).$$

   Or if that notation is too ambiguous for you, maybe call it

   $$\varphi(a \bmod 6) = (a \bmod 2,\ a \bmod 3).$$

   Write out what φ does to $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ explicitly. Notice that it's a bijection.

   Convince yourselves that φ respects addition, multiplication, and 1, so it's an isomorphism.

3. Convince yourselves that there is *no* isomorphism $\varphi \colon \mathbb{Z}_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2$, because if φ respects addition then $\varphi(\bar{3})$ has to equal $\varphi(\bar{1})$, so φ can't be a bijection.

In general, the Chinese Remainder Theorem can be interpreted as saying that if $\gcd(m, n) = 1$ then the map

$$\begin{aligned} \mathbb{Z}_{mn} &\to \mathbb{Z}_m \times \mathbb{Z}_n \\ \bar{a} &\mapsto \quad (\bar{a},\, \bar{a}) \end{aligned}$$

is an isomorphism.