Let $R$ and $S$ be rings. As a set, the Cartesian product $R \times S$ consists of pairs $(r, s)$ where $r \in R$ and $s \in S$. Put a ring structure on $R \times S$ by defining

$$(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$$
$$(r_1, s_1) \cdot (r_2, s_2) = (r_1 \cdot r_2, s_1 \cdot s_2).$$

You can check, although it’s kind of tedious, that this definition of $+$ and $\cdot$ satisfies the ring axioms.

1. What is the additive identity in $R \times S$?
   What is the multiplicative identity?
2. Let $\varphi: \mathbb{Z}_6 \to \mathbb{Z}_2 \times \mathbb{Z}_3$ be the map given by
   $$\varphi(\bar{a}) = (\bar{a}, \bar{a}).$$
   Or if that notation is too ambiguous for you, maybe call it
   $$\varphi(a \mod 6) = (a \mod 2, a \mod 3).$$
   Write out what $\varphi$ does to \{0, 1, 2, 3, 4, 5\} explicitly. Notice that it’s a bijection.
   Convince yourselves that $\varphi$ respects addition, multiplication, and 1, so it’s an isomorphism.
3. Convince yourselves that there is no isomorphism $\varphi: \mathbb{Z}_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2$,
   because if $\varphi$ respects addition then $\varphi(3)$ has to equal $\varphi(1)$, so $\varphi$ can’t be a bijection.

In general, the Chinese Remainder Theorem can be interpreted as saying that if $\gcd(m, n) = 1$ then the map

$$\mathbb{Z}_{mn} \to \mathbb{Z}_m \times \mathbb{Z}_n$$
$$\bar{a} \mapsto (\bar{a}, \bar{a})$$

is an isomorphism.