The Gaussian integers $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ look like this:

\[
\begin{array}{cccc}
\ldots & \ldots & i & \ldots \\
\ldots & \ldots & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\end{array}
\]

1. Show that the map $\varphi: \mathbb{Z}[i] \rightarrow \mathbb{Z}_2$ given by $\varphi(a + bi) = \bar{a} + \bar{b}$ is a homomorphism. This means:
   (a) $\varphi(1) = \bar{1}$,
   (b) $\varphi(z + w) = \varphi(z) + \varphi(w)$ for all $z, w \in \mathbb{Z}[i]$, and
   (c) $\varphi(z \cdot w) = \varphi(z) \cdot \varphi(w)$ for all $z, w \in \mathbb{Z}[i]$.

2. Show that the map $\psi: \mathbb{Z}[i] \rightarrow \mathbb{Z}_2$ given by $\psi(a + bi) = \bar{a}$ is not a homomorphism.
   Hint: We see that $\psi(1) = \bar{1}$, so you need to find two elements $z, w \in \mathbb{Z}[i]$ such that either
   \[
   \psi(z + w) \neq \psi(z) + \psi(w),
   \]
   or
   \[
   \psi(z \cdot w) \neq \psi(z) \cdot \psi(w).
   \]

3. If you have time, take the picture above and color the points that belong to $\ker(\varphi)$, that is, the points $z \in \mathbb{Z}[i]$ with $\varphi(z) = \bar{0}$.
   Challenge: Show that this is exactly the set of all multiples of $1 + i$. 
