an ideal is a non-empty subset $I \subset R$ such that...

(last worksheet: let $R$ be a comm. ring
let $I, J \subset R$ be ideals.

(examples: $R = \mathbb{Z}$, $I = \langle 6 \rangle$ or $\langle 10 \rangle$
$R = \mathbb{Q}[x]$, $I = \langle x^2 + x \rangle$ or $\langle x^2 + 1 \rangle$
$R = \mathbb{Q}[\sqrt{5}]$, $I = \langle 2 \rangle$ or $\langle 2, 1 + \sqrt{5} \rangle$)

Prop: $I \cap J$ is an ideal.

Pf: let $a, b \in I \cap J$ and $r \in R$

Then $a + b \in I$ and $ra \in I$ (b.c. $I$ is an ideal.
$a + b \in J$ and $ra \in J$ (b.c. $J$ is an ideal
so $a + b \in I \cap J$ and $ra \in I \cap J$

(examples: in $\mathbb{Z}$, $\langle 6 \rangle \cap \langle 10 \rangle = \langle 30 \rangle$

in $\mathbb{Q}[x]$, $\langle x^2 + x \rangle \cap \langle x^2 + 1 \rangle = \langle x^4 + x^3 + x^2 + x \rangle$

$\cap$ is like least common multiple (lcm)

Next: defined $I + J = \{a + b \mid a \in I, b \in J\}$

Prop: $I + J$ is an ideal.

Pf: take two elts. of $I + J$,
call them $a + b$ and $a' + b'$
where $a, a' \in I, b, b' \in J$
Then \((a + b) + (a' + b') = (a + a') + (b + b') \in I + J\)

and for \(r \in \mathbb{R}\), have \(r(a+b) = r a + r b \in I + J\) \(\square\)

**Example:** \(<6> + <10> \subseteq \mathbb{Z} \prod\)

That's \(<6,10> = <2>\)

\(<6> = \{ 6m \mid m \in \mathbb{Z} \}\)

\(<10> = \{ 10n \mid n \in \mathbb{Z} \}\)

\(<6> + <10> = \{ 6m + 10n \mid m, n \in \mathbb{Z} \}\) that's \(<6,10> \prod\)

by def.

= \(<2>\) as we've seen

\(I + J\) is like greatest common divisor \((\text{gcd})\)

\(<6> + <11> = <1>\)

Notice: In any ring, \(<1> = \{ r \cdot 1 \mid r \in R \}\) = the whole ring \(R\).

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Product of ideals.

might want to define \(I \cdot J = \{ ab \mid a \in I, b \in J \}\) \(\text{not!}\)

but that might not be closed under +

Instead, define \(I \cdot J = \text{sums of } a \cdot b\)'s

= \(\{ a_1 b_1 + \cdots + a_k b_k \mid a_i \in I, b_i \in J \text{ \(k\) is arbitrary} \} \)
Prop: \( I \cdot J \) is an ideal.

Pf: take two elements of \( I \cdot J \)
write them as
\[ a_1 b_1 + \cdots + a_k b_k \quad \text{and} \quad c_1 d_1 + \cdots + c_l d_l \]
when \( a_i, c_i \in I \) and \( b_i, d_i \in J \)

then \( a_1 b_1 + \cdots + a_k b_k + c_1 d_1 + \cdots + c_l d_l \in I \cdot J \)

and \[ r \cdot (a_1 b_1 + \cdots + a_k b_k) = (ra_1) b_1 + \cdots + (ra_k) b_k \in I \cdot J \]

\[ \square \]

Example: \( I = \langle 6 \rangle \subset \mathbb{Z} \quad J = \langle 10 \rangle \subset \mathbb{Z} \)

claim that \( I \cdot J = \langle 60 \rangle \)

proof: take some element \( 6m \in I \)

and some \( 10n \in J \)

then \( 6m \cdot 10n = 60mn \in \langle 60 \rangle \)

and any sum of terms like that stays in \( \langle 60 \rangle \)

\[ \sum 6m_i \cdot 10n_i = \sum 60m_i n_i \text{ still in } \langle 60 \rangle \]

so \( I \cdot J = \langle 60 \rangle \).

070H, \( 60 = 6 \cdot 10 \in I \cdot J \)

so by Friday's homework, \( \langle 60 \rangle \subset I \cdot J \)

thus \( I \cdot J = \langle 60 \rangle \).
Also: \( I = \langle 6 \rangle \), \( I \cdot I = \langle 36 \rangle \).

**Prop:** if \( R \) is comm.
\[
I = \langle a \rangle \quad \text{and} \quad J = \langle b \rangle
\]
then \( I \cdot J = \langle ab \rangle \)

**Proof:** almost identical to what I wrote for \( \langle 6 \rangle \cdot \langle 10 \rangle \) in \( \mathbb{Z} \).

**Worksheet:** \( R = \mathbb{Z}[\sqrt{5}] \)
\[
I = \langle 2, 1+\sqrt{5} \rangle \quad \text{two more} \ J, \ K, \ \text{study} \ I \cdot J \ \text{etc.}
\]

**Next time:** if \( I + J = \langle 1 \rangle \) \( \quad \text{in any comm.}
\]
then \( I \cdot J = I \cap J \) \( \quad \text{in any ring.}
\]
**Analogy:** if \( \gcd(a, b) = 1 \) \( \quad \text{in \( \mathbb{Z} \) or \( \mathbb{Q}[x] \)}
\]
then \( a \cdot b = (\text{lcm}(a, b)) \)