an ideal is a <u>non-empty</u> subset $I \subset R$ such that...

last worksheet: let R be a comm. ring
let $I, J \subset R$ be ideals.

(examples: $R = \mathbb{Z}$ $\quad I = \langle 6 \rangle$ or $\langle 10 \rangle$
$R = \mathbb{Q}[x]$ $\quad I = \langle x^2 + x \rangle$ or $\langle x^2 + 1 \rangle$
$R = \mathbb{Z}[\sqrt{-5}]$ $\quad I = \langle 2 \rangle$ or $\langle 2, 1 + \sqrt{-5} \rangle$)

<u>Prop</u>: $I \cap J$ is an ideal.

<u>Pf</u>: let $a, b \in I \cap J$ and $r \in R$

then $a + b \in I$ and $ra \in I$ bec. I is an ideal.
$a + b \in J$ and $ra \in J$ " J " " " "
So $a + b \in I \cap J$ and $ra \in I \cap J$ ∎

examples: in $\mathbb{Z}$, $\langle 6 \rangle \cap \langle 10 \rangle = \langle 30 \rangle$

in $\mathbb{Q}[x]$, $\underbrace{\langle x^2 + x \rangle}_{= x(x+1)} \cap \langle x^2 + 1 \rangle = \langle x^4 + x^3 + x^2 + x \rangle$

$\cap$ is like least common multiple (lcm)

Next: defined $I + J = \{ a + b \mid a \in I, b \in J \}$

Prop: $I + J$ is an ideal.
Pf: take two elts. of $I + J$,

call them $a + b$ and $a' + b'$
where $a, a' \in I$ $\quad b, b' \in J$

then $(a+b)+(a'+b') = (\underline{a+a'}) + (\underline{b+b'}) \in I+J$

and for $r \in R$, have $r(a+b) = \underset{\substack{\cap \\ I}}{\underline{ra}} + \underset{\substack{\cap \\ J}}{\underline{rb}} \in I+J$ $\boxed{\checkmark}$

example: $\langle 6\rangle + \langle 10\rangle$ in $\mathbb{Z}$
that's $\langle 6, 10\rangle = \langle 2\rangle$

$\langle 6\rangle = \{ 6m \mid m \in \mathbb{Z}\}$
$\langle 10\rangle = \{ 10n \mid n \in \mathbb{Z}\}$

$\langle 6\rangle + \langle 10\rangle = \{ 6m + 10n \mid m, n \in \mathbb{Z}\}$ that's $\langle 6, 10\rangle$ by def.

$= \langle 2\rangle$ as we've seen

$I+J$ is like greatest common divisor (gcd)

$\langle 6\rangle + \langle 11\rangle = \langle 1\rangle$

notice: in any ring, $\langle 1\rangle = \{ r \cdot 1 \mid \text{any } r \in R\}$
$=$ the whole ring R.

products of ideals.
might want to define $I \cdot J = \{ ab \mid a \in I, b \in J\}$
no!

but that might not be closed under +

instead, define $I \cdot J$ = sums of $a \cdot b$'s
$= \{ a_1b_1 + \cdots + a_kb_k \mid a_i \in I, b_i \in J$
k is arbitrary. $\}$

Prop: $I \cdot J$ is an ideal.

Pf: take two elements of $I \cdot J$
write them as
$a_1 b_1 + \cdots + a_k b_k$ and $c_1 d_1 + \cdots + c_\ell d_\ell$
where $a_i, c_i \in I$ and $b_i, d_i \in J$

then $a_1 b_1 + \cdots + a_k b_k + c_1 d_1 + \cdots + c_\ell d_\ell \in I \cdot J$

and $r \cdot (a_1 b_1 + \cdots + a_k b_k) = (\underbrace{r a_1}_{\in I}) \underbrace{b_1}_{\in J} + \cdots + (r a_k) b_k \in I \cdot J$

$\boxed{\checkmark}$

Example: $I = \langle 6 \rangle \subset \mathbb{Z}$ $\quad J = \langle 10 \rangle \subset \mathbb{Z}$
claim that $I \cdot J = \langle 60 \rangle$

proof: take some element $6m \in I$
and some $10n \in J$
then $6m \cdot 10n = 60 \cdot mn \in \langle 60 \rangle$

and any sum of terms like that stays in $\langle 60 \rangle$

$$\sum_{i=1}^{k} 6m_i \cdot 10n_i = \sum 60 \cdot m_i n_i \text{ still in } \langle 60 \rangle$$

So $I \cdot J \subset \langle 60 \rangle$.

6 ⇝ a
10 ⇝ b
$m, n \in R$
rather than $\mathbb{Z}$

OTOH, $60 = 6 \cdot 10 \in I \cdot J$

so by Friday's homework, $\langle 60 \rangle \subset I \cdot J$

thus $I \cdot J = \langle 60 \rangle$.

$\boxed{\checkmark}$

Also: $I = \langle 6 \rangle$. $I \cdot I = \langle 36 \rangle$.

Prop: if R is comm.,
$I = \langle a \rangle$ and $J = \langle b \rangle$
then $I \cdot J = \langle ab \rangle$

Proof: almost identical to what I wrote for $\langle 6 \rangle \cdot \langle 10 \rangle$ in $\mathbb{Z}$. $\boxed{\text{Q}}$

Worksheet: $R = \mathbb{Z}[\sqrt{-5}]$

$I = \langle 2, 1+\sqrt{-5} \rangle$ two more J, K,

study $I \cdot J$ etc.

Next time: if $I + J = \langle 1 \rangle$
then $I \cdot J = I \cap J$ (in any comm. ring.)

analogy: if $\gcd(a,b) = 1$
then $a \cdot b = \operatorname{lcm}(a,b)$ (in $\mathbb{Z}$ or $\mathbb{Q}[x]$