$\phi : R \longrightarrow S$ a "ring homomorphism"
or just a "homomorphism"

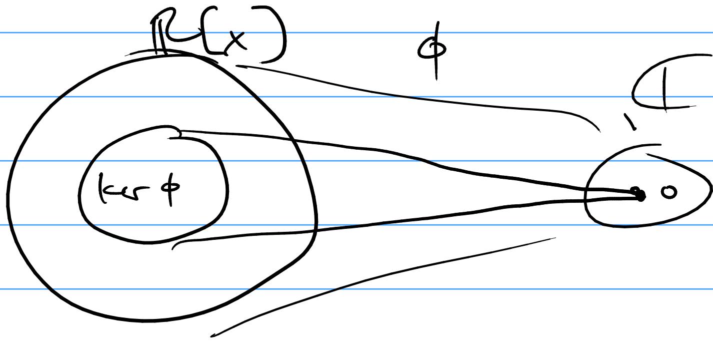
evaluation map!, aka "plugging in"

e.g. $\phi : \mathbb{R}[x] \longrightarrow \mathbb{C}$

given by $\phi(f) = f(i)$

$f(x^2+2x+3) = i^2+2i+3 = 2i+2$

$+$ and $\times$ on $\mathbb{R}[x]$ are cooked up
to make ϕ a homomorphism.



in this example,
$\ker\phi = \langle x^2+1 \rangle$

if $f \in \mathbb{R}[x]$
if $f(i)=0$ then $x-i \mid f$ in $\mathbb{C}[x]$

but $f(-i) = \overline{f(i)}$ because coeffs of f are real.
So $f(-i)=0$, so $x+i \mid f$

$\gcd(x-i, x+i) = 1$ so
$(x+i)(x-i) \mid f$

so $x^2+1 \mid f$ in $\mathbb{C}[x]$
so $x^2+1 \mid f$ in $\mathbb{R}[x]$ by long division.

so $\ker\phi \subset \langle x^2+1\rangle$

also $\phi(x^2+1) = i^2+1 = 0$

so $x^2+1 \in \ker\phi$

so $\langle x^2+1\rangle \subset \ker\phi$

recall $\phi : \mathbb{R}[x] \longrightarrow \mathbb{C}$

$f \longmapsto f(i)$

last week, saw that $\mathbb{R}[x]/\langle x^2+1\rangle$ looks "the same" as $\mathbb{C}$

=

on HW 2, saw a hom

$\psi : \mathbb{Z}[\sqrt{-5}] \twoheadrightarrow \mathbb{Z}_3$

$a+b\sqrt{-5} \longmapsto \bar{a}-\bar{b}$

$\ker\psi = \langle 3, 1+\sqrt{-5}\rangle$

$\mathbb{Z}[\sqrt{-5}]/\langle 3, 1+\sqrt{-5}\rangle$ looks "the same as" $\mathbb{Z}_3$

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Noether's first iso thm.

let $\phi : R \to S$ be a (ring) homomorphism

recall that $\ker\phi \subset R$ is an ideal.

and $\operatorname{im}\phi \subset S$ is a subring.

then $R/\ker\phi \cong \operatorname{im}\phi$. isomorphic

(if ϕ is surjective then $\operatorname{im}\phi = S$, so $R/\ker\phi \cong S$)

Proof: given an element of $R/\ker\phi$, write it as $\bar{r}$ for some $r \in R$.
define a map $\psi: R/\ker\phi \to \operatorname{im}\phi$
by setting $\psi(\bar{r}) = \phi(r)$
Claim that ψ is an isomorphism

① ψ is well-defined.
if $\bar{r} = \bar{s}$, want $\phi(r) = \phi(s)$

if $\bar{r} = \bar{s}$ then $r \equiv s \mod \ker\phi$
so $r - s \in \ker\phi$
so $\phi(r-s) = 0$
so $\phi(r) - \phi(s) = 0$
so $\phi(r) = \phi(s)$. ✓

② ψ is surjective.
given $s \in \operatorname{im}\varphi \subset S$
write $s = \varphi(r)$
then $s = \psi(\bar{r})$ ✓

③ ψ is a homomorphism.

$\psi(\bar{1}) = \varphi(1) = 1$

$$
\begin{aligned}
\psi(\bar{r} + \bar{s}) &= \psi(\overline{r+s}) \\
&= \varphi(r+s) \\
&= \varphi(r) + \varphi(s) \\
&= \psi(\bar{r}) + \psi(\bar{s})
\end{aligned}
$$

similarly $\psi(\bar{r} \cdot \bar{s}) = \psi(\bar{r}) \cdot \psi(\bar{s})$ ✓

(4) ψ is injective.
By HW enough to show that
$\ker \psi = \langle \bar{0} \rangle$.

if $\psi(\bar{r}) = 0$
then $\varphi(r) = 0$
so $r \in \ker \phi$
so $r \equiv 0 \mod \ker \phi$
so $\bar{r} = \bar{0}$ ✓

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