Last time: the first iso. thm.

a hom. $\varphi: R \longrightarrow S$

induces an iso $\psi: R/\underbrace{\ker\varphi}_{\ker\varphi \text{ is an ideal in } R} \longrightarrow \underbrace{\operatorname{im}\varphi}_{\text{image of } \varphi \text{ is a subring of } S.}$

Today's homework #1c,

$$\phi: \mathbb{Q}[x] \longrightarrow \mathbb{Q} \times \mathbb{Q}$$
$$f \longmapsto (f(5),\ f(6))$$

surjective, homomorphism, $\ker\phi = \langle x^2 - 11x + 30 \rangle$

thus $\mathbb{Q}[x] / \langle x^2 - 11x + 30 \rangle \cong \mathbb{Q} \times \mathbb{Q}$

Similar but simpler:

$$\phi: \mathbb{Q}[x] \longrightarrow \mathbb{Q}$$
$$f \longmapsto f(5)$$

"evaluate f at 5"

surjective: given $a \in \mathbb{Q}$,
take $f =$ const' polynomial a.
then $\phi(f) = a$.

kernel? $f \in \ker\phi$
iff $\phi(f) = 0$
iff $f(5) = 0$
iff $x - 5 \mid f$ (root-factor theorem)
iff $f \in \langle x - 5 \rangle$

So $\mathbb{Q}[x] / \langle x-5 \rangle \cong \mathbb{Q}$

Another example: $\mathbb{Z}[x]/\langle 2x-1\rangle$.

in quot. ring, $\overline{2x-1} = \bar{0}$

so $\bar{2}\bar{x} = \bar{1}$ so $\bar{x}$ is like $1/2$ ...

So $\bar{x}^2$ is like $1/4$?

↳ in the sense that $\bar{4}\cdot\bar{x}^2 = (\bar{2}\bar{x})^2 = \bar{1}^2 = \bar{1}$

So $\bar{x}^2 + \bar{5}\bar{x} + \bar{6}$ is like $\frac{1}{4} + \frac{5}{2} + 6 = 35/4$

↳ how? $\bar{4}\cdot(\bar{x}^2 + \bar{5}\bar{x} + \bar{6}) = \bar{4}\bar{x}^2 + \overline{20}\bar{x} + \overline{24}$

$= \bar{1} + \overline{10} + \overline{24} = \overline{35}$

$\mathbb{Z}[x]/\langle 2x-1\rangle$ looks like $\mathbb{Z}[\frac{1}{2}] = \left\{ \frac{m}{n} \in \mathbb{Q} \;\middle|\; \begin{matrix} n \text{ is a} \\ \text{power of } 2 \end{matrix} \right\} \subset \mathbb{Q}$

1st iso thm says to look at the hom

$$\mathbb{Z}[x] \xrightarrow{\phi} \mathbb{Q}$$
$$f \longmapsto f(1/2)$$

image of ϕ is $\mathbb{Z}[1/2] \subset \mathbb{Q}$

given some $\frac{a}{2^k} \in \mathbb{Z}[1/2]$, take $f = a\cdot x^k \in \mathbb{Z}[x]$

then $\phi(f) = a\cdot(\frac{1}{2})^k$ ✓

is $\ker\phi = \langle 2x-1\rangle$?

$\phi(2x-1) = 2\cdot\frac{1}{2} - 1 = 0$ so $2x-1 \in \ker\phi$ so $\langle 2x-1\rangle \subset \ker\phi$

conversely, if $f(\frac{1}{2}) = 0$

root-factor thm says $f = (x - \frac{1}{2}) \cdot g$ for some $g \in \mathbb{Q}[x]$

want: $f = (2x-1) \cdot h$ for some $h \in \mathbb{Z}[x]$.

Turns out to be true... mess with Gauss's lemma or details of long division...

Worksheet: #1, want $\mathbb{Z}[i] / \langle 1+i \rangle \cong \mathbb{Z}_2$

so want a hom $\varphi : \mathbb{Z}[i] \longrightarrow \mathbb{Z}_2$

with $\ker \varphi = \langle 1+i \rangle$

Old worksheet: try $\varphi(a+bi) = \bar{a} + \bar{b}$
$= \bar{a} - \bar{b}$

On #2, question about why we need $\bar{a} \in \mathbb{Z}_{10}$ with $\bar{a}^2 = -\bar{1}$?

Related: Suppose I want a hom $\varphi : \mathbb{Z}[\sqrt{-5}] \longrightarrow \mathbb{Z}_7$

must have $\varphi(1) = \bar{1}$

so $\varphi(2) = \varphi(1+1) = \varphi(1) + \varphi(1) = \bar{1} + \bar{1} = \bar{2}$

and $\varphi(3) = \bar{3}$ and so on.

must also have $\varphi(\sqrt{-5}) \cdot \varphi(\sqrt{-5}) = \varphi(\sqrt{-5} \cdot \sqrt{-5})$

$= \varphi(-5)$

$= -\bar{5} = \bar{2}$

in $\mathbb{Z}_7$,

a	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$
a^2	$\bar{0}$	$\bar{1}$	$\bar{4}$	$\bar{2}$	$\bar{2}$	$\bar{4}$	$\bar{1}$

So $\varphi(\sqrt{-5}) = \bar{3}$ or $\bar{4}$

then $\varphi(a+b\sqrt{-5}) = \bar{a} + \bar{3}\bar{b}$ or $\bar{a} + \bar{4}\bar{b}$

Check: either of those is a homomorphism.

Continue on same worksheet.

Challenge: $\mathbb{Z}[i]/\langle 2\rangle \not\cong \mathbb{Z}_4$

also $\not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$