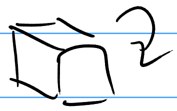
Worksheet: rotations of 

One asked: is there some rotation where by doing it repeatedly we get every rotation?

↳ rotating one face by 90° doesn't work, but maybe something fancier?

want $g \in G$ such that $1, g, g^2, g^3, \ldots$

gives every element of G.

$g^0 = e = 1$ identity element

$$\langle g \rangle = \{\ldots, g^{-2}, g^{-1}, 1, g, g^2, g^3, \ldots\}$$

is called the <u>cyclic subgroup</u> generated by g.

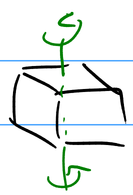
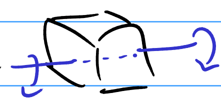
If there were such a $g \in G$, then G would be Abelian:

$$g^m \cdot g^n = (\overbrace{g \cdot g \cdots g}^{m}) \cdot (\overbrace{g \cdots g}^{n \text{ times}})$$

$$= g^{m+n}$$

$$= g^n \cdot g^m$$

But this group is not Abelian:

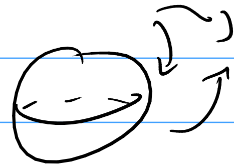
if $h =$  $k =$ 

then $hk \neq k \cdot h$

Def the *order* of a group G is the # of elements, or ∞ if the group is infinite.

notation: $|G|$

we'll mostly study finite groups, but infinite groups are also cool:

e.g. rotations of a sphere 

Def. the *order* of an element $g \in G$ is the smallest positive integer n such that $g^n = 1$ or ∞ if there is no such n.

Prop if $|G|$ is finite, then $|g|$ is finite for every $g \in G$

Proof consider $g, g^2, g^3, \ldots$

if G is finite, they can't all be different.

so $g^m = g^n$ for some $m \neq n$

if $m < n$, multiply by $g^{-m} = (g^{-1})^m$

$$g^{-m} \cdot g^m = g^{-m} \cdot g^n$$

$$1 = g^{n-m} \quad \text{and} \quad n-m > 0$$

if $m > n$, it's similar.

Observation: the order of the element $g \in G$ is the order of the subgroup $\langle g \rangle \subset G$.

Back to G = rotations of a cube.

Saw that $|G| = 24$.

take your favorite face and do a rotation...
6 places it could end up,
4 ways to rotate once it gets there.

or take your fav. vertex
8 places it could go
· 3 orientations once it gets there.

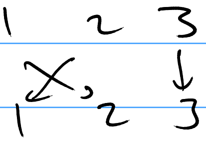
or 12 edges · 2 orientations for an edge

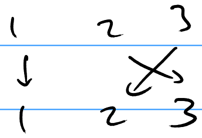
Introduced the <u>symmetric group</u> or <u>permutation group</u>

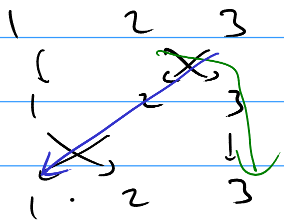
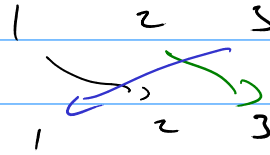
$S_n = \{$ bijections from $\{1, 2, \ldots, n\}$ to itself. $\}$

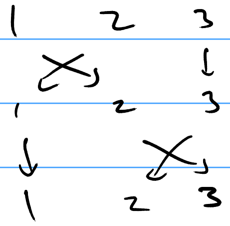
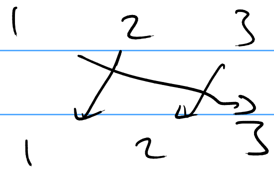
$|S_n|$ = (n places that 1 could go)
· (($n-1$) places that 2 could go)
· (($n-2$) places that 3 could go)

$= n!$

Not Abelian: if σ is 

and τ is 

then $\sigma \circ \tau$ is  = 

whereas $\tau \circ \sigma$ is  = 

✓

if $G =$ rotations of a cube
and we label the faces 1, 2, 3, 4, 5, 6

then we get a homomorphism

$$G \hookrightarrow S_6$$

$$g \longmapsto \text{how does permute the 6 faces?}$$

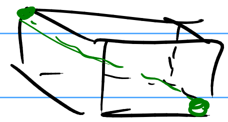
injective, not surjective

$$24 \hookrightarrow 6! = 720$$

label vertices 1 ... 8,
get a hom. $G \hookrightarrow S_8$

label edges 1 ... 12
get a hom. $G \hookrightarrow S_{12}$

fun: 4 "long diagonals"



$G \longrightarrow S_4$ is it an isomorphism?
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Worksheet: analyze how many elements of each order in G.