Worksheet: Dihedral groups

\[ D_5 \text{ = symms. of a pentagon} \]

\[ r = \text{rotation} \]

\[ r^5 = 1 \]

\[ s = \text{reflection} \]

\[ s^2 = 1 \]

\[ rs = sr^{-1} = sr^{-1} \]

\[ r^3 s = r rs = rs r^{-1} = s r^{-1} r^{-1} = sr^2 = s \]

\[ r^3 s = sr^{-3} = sr^2 \]

\[ r^4 s = sr^{-4} = sr \]

Typical of dihedral groups:

\[ D_n = \left\{ 1, r, r^2, \ldots, r^{n-1} \right\} \quad r^n = 1 \]

\[ s, sr, sr^2, \ldots, sr^{n-1} \]

\[ rs = sr^{-1} \]

==

Group Actions

Def. A left action of a group \( G \) on a set \( X \)

is a map \( G \times X \rightarrow X \)

\[ (g, x) \mapsto g \cdot x \]

such that

1. \( g \cdot (h \cdot x) = (g \cdot h) \cdot x \quad \text{for all } g, h \in G \)
2. \( 1 \cdot x = x \)
Could also do right actions...

Examples:

1A. \( G = \text{rotation of a cube} \), \( X = \{ \text{the 6 faces of the cube} \} \)

If \( g \) is some rotation and \( x \) is some face, then \( g \cdot x = \) where that face ends up.

1B. \( G = \text{same} \)

\( X = \{ \text{the 8 vertices} \} \)

1C. \( G = \text{same} \)

\( X = \{ \text{the 12 edges} \} \)

2A. \( G = \text{D}_4 \), \( X = \{ \text{vertices of a square} \} \)

\( r \cdot (1) = 2 \)

\( r^2 \cdot (1) = 3 \)

\( s_r \cdot (1) = s \cdot (r \cdot (1)) = s \cdot 2 = 1 \)

orbit of \( 1 \) is \( \{ 1, 2, 3, 4 \} \)

2B. \( G = \text{D}_4 \), \( X = \{ \text{edges of a square} \} \)

\( r \cdot a = b \)

\( r^2 \cdot a = c \)

\( s_r \cdot a = a \quad \text{or} \quad s_r \cdot a = s \cdot (r \cdot a) = s \cdot b = d \)

orbit of \( a \) is \( \{ a, b, c, d \} \)

orbit of \( a \) is \( \{ 1, 5 \} \)
\( G = (\mathbb{R}_+^1, +) \)

\[ X = \mathbb{C} \]

for \( t \in \mathbb{R}, \quad z \in \mathbb{C} \), define

\[ t \cdot z = e^{it}z \]

\( t \) rotates \( \mathbb{C} \) by that many radians clockwise.

satisfies the 2 axioms for an action.

for \( s, t \in \mathbb{R} \) and \( z \in \mathbb{C} \)

\[ s \cdot (t \cdot z) = s \cdot (e^{it}z) = e^{is}e^{it}z \]

\[ = e^{i(s+t)}z = (s+t) \cdot z \]

\[ 0 \cdot z = e^{i0}z = 1z = z \]

if you prefer \( (x,y) \in \mathbb{R}^2 \) rather than \( z = x + iy \in \mathbb{C} \),
could write \( t \cdot (x,y) = (\cos t \cdot x - \sin t \cdot y, \sin t \cdot x + \cos t \cdot y) \)

use angle addition formulas to

see that \( s \cdot (t \cdot (x,y)) = (s+t) \cdot (x,y) \)
still $G = \mathbb{R}$ under addition
\[ x = 0 \]

now for $t \in \mathbb{R}$ and $z \in \mathbb{C}$
define $t \cdot z = e^{t} z$

Let a group $G$ act on a set $X$.

**Def.** for an element $x \in X$,
the **orbit** of $x$ is
\[ O_{x} = \{ g \cdot x \mid g \in G \} \subset X \]

**Def.** the **stabilizer** of $x$ is
\[ G_{x} = \{ g \in G \mid g \cdot x = x \} \subset G \]

**Thm:** $|O_{x}| \cdot |G_{x}| = |G|$