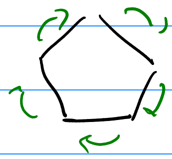
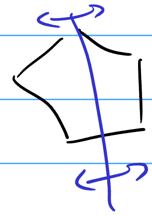
Worksheet: Dihedral groups...

D_5 = symms. of a pentagon

$r =$  $s =$ 

$r^5 = 1$ $s^2 = 1$

$rs = sr^4 = sr^{-1}$

$r^2 s = r\,rs = rs\,r^{-1} = sr^{-1}r^{-1} = sr^{-2} = sr^3$

$r^3 s = sr^{-3} = sr^2$

$r^4 s = sr^{-4} = sr$

typical of dihedral groups:

$$D_n = \left\{ \begin{matrix} 1, & r, & r^2, & \dots & r^{n-1} \\ s, & sr, & sr^2, & \dots, & sr^{n-1} \end{matrix} \right\} \qquad \begin{matrix} r^n = 1 \\ s^2 = 1 \\ rs = sr^{-1} \end{matrix}$$

Group Actions

Def A left action of a group G on a set X is a map $G \times X \to X$

$$(g, x) \longmapsto g \cdot x$$

such that

$\cdot$ = G acting on X

$\cdot$ = group op. in G

① $g \cdot (h \cdot x) = (g \cdot h) \cdot x$

② $1 \cdot x = x$

Could also do right actions...

Examples:

(1A) $G =$ rotations of a cube, $X = \{$the 6 faces of the cube$\}$

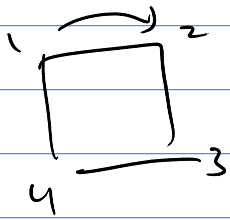
orbit of one face = all of X

if g is some rotation and $x =$ some face then $g \cdot x =$ where that face ends up.

(1B) $G =$ same
$X = \{$the 8 vertices$\}$

(1C) $G =$ same
$X = \{$the 12 edges$\}$

(2A) $G = D_4$ $\quad X = \{$vertices of a square$\}$



$r \cdot 1 = 2$

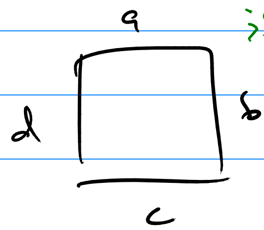
$r^2 \cdot 1 = 3$

orbit of 1 is $\{1,2,3,4\}$

$sr \cdot 1 = s \cdot (r \cdot 1) = s \cdot 2 = 1$

(2B) $G = D_4$ $\quad X = \{$edges of a square$\}$

orbit of a is $\{a,b,c,d\}$



$r \cdot a = b$

$r^2 \cdot a = c$

stab. of a is $\{1, s\}$

$s \cdot a = a \qquad sr \cdot a = s \cdot (r \cdot a) = s \cdot b = d$

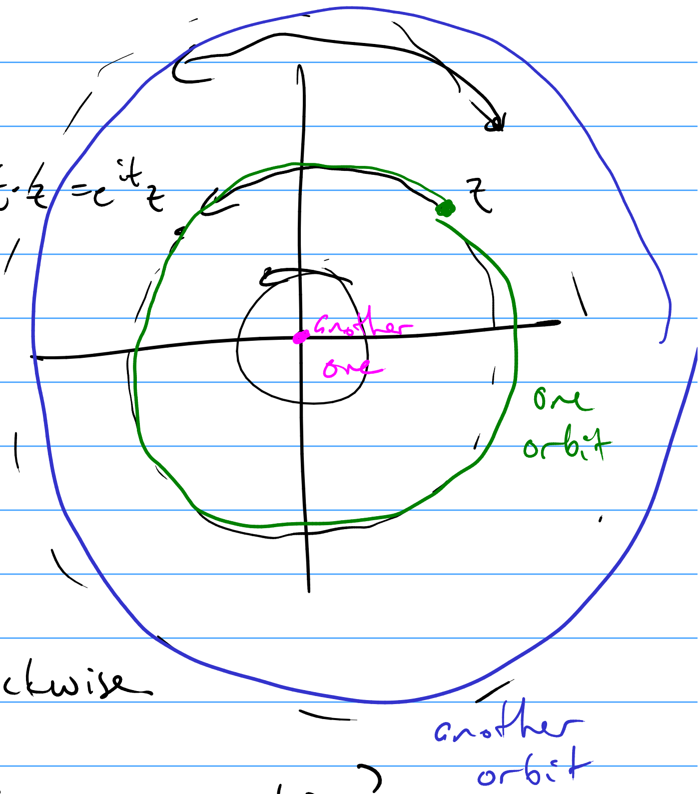
3A $\quad G = (\mathbb{R}, +)$

$X = \mathbb{C}$

for $t \in \mathbb{R}$, $z \in \mathbb{C}$, define

$$t \cdot z = e^{it} z$$

t rotates $\mathbb{C}$ by that many radians clockwise



satisfies the 2 axioms for an action?

for $s, t \in \mathbb{R}$ and $z \in \mathbb{C}$

$$s \cdot (t \cdot z) = s \cdot (e^{it} z) = e^{is} e^{it} z$$

$$= e^{i(s+t)} z = (s+t) \cdot z \quad \checkmark$$

$$0 \cdot z = e^{i0} z = 1z = z \quad \checkmark$$

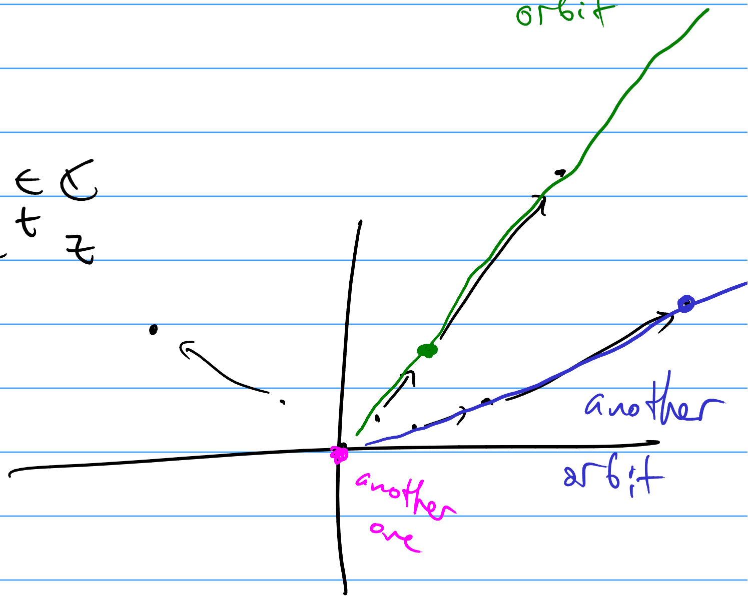
if you prefer $(x,y) \in \mathbb{R}^2$ rather than $z = x + iy \in \mathbb{C}$,

could write $t \cdot (x,y) = (\cos t \, x - \sin t \, y, \; \sin t \, x + \cos t \, y)$

use angle addition formulas to see that $s \cdot (t \cdot (x,y)) = (s+t) \cdot (x,y)$

(3B) still $G = \mathbb{R}$ under addition
$X = \mathbb{C}$

now for $t \in \mathbb{R}$ and $z \in \mathbb{C}$
define $t \cdot z = e^t z$



Let a group G act on a set X.

Def. for an element $x \in X$, the orbit of x is

$$\mathcal{O}_x = \{ g \cdot x \mid g \in G \} \subset X$$

Some call it Gx

Def. the stabilizer of x is

$$G_x = \{ g \in G \mid g \cdot x = x \} \subset G$$

Some call it $\mathrm{Stab}(x)$

coming soon:

Thm: $|\mathcal{O}_x| \cdot |G_x| = |G|$