Last time: a (left) action of a group G on a set X.

needs to satisfy $g\cdot(h\cdot x) = (gh)\cdot x$
and $1\cdot x = x$

The <u>orbit</u> of an element $x\in X$ is

$$\mathcal{O}_x = \{\, g\cdot x \mid g\in G \,\} \subset X$$

The <u>stabilizer</u> of x is

$$G_x = \{\, g \mid gx = x \,\} \subset G$$

it's a subgroup: if $g, h \in G_x$
then $(gh)\cdot x = g\cdot(h\cdot x) = g\cdot x = x$
so $gh \in G_x$

and $g^{-1}\cdot x = g^{-1}\cdot(g\cdot x) = (g g^{-1})\cdot x$
$= 1\cdot x = x$

so $g^{-1} \in G_x$ $\boxed{\varepsilon}$

Soon: the orbit-stabilizer theorem:

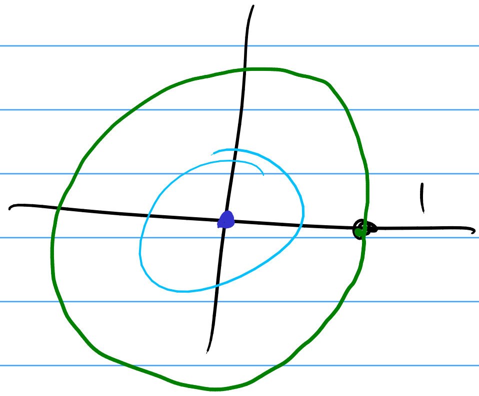
$$|\mathcal{O}_x|\cdot|G_x| = |G|$$

$$\left(\#\text{ elements in } \mathcal{O}_x \cdot \#\text{ elements in } G_x = \#\text{ elements in } G.\right)$$

Examples: $\mathbb{R}$ acts on $\mathbb{C}$ by rotations

$$t \cdot z = e^{it} z$$

orbit of $1 \in \mathbb{C}$
is $\{ e^{it} \cdot 1 \mid t \in \mathbb{R} \}$
$=$ unit circle



stabilizer of 1 is
$\{ t \in \mathbb{R} \mid e^{it} = 1 \}$

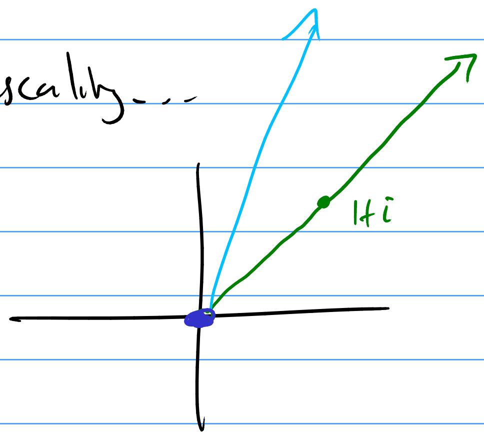
$$= \{ 2\pi n \mid n \in \mathbb{Z} \} = \{ 0, \pm 2\pi, \pm 4\pi, \dots \}$$

subgroup of $\mathbb{R}$ under addition.

orbit of 0 is $\{0\}$ $\qquad e^{it} \cdot 0 = 0$
stabilizer is all of $\mathbb{R}$ $\qquad \forall t$

example: $\mathbb{R}$ acts on $\mathbb{C}$ by rescaling...

$$t \cdot z = e^{t} z$$



orbits are rays
orbit of $1+i$ is
$\{ e^{t} \cdot (1+i) \mid t \in \mathbb{R} \}$

stabilizer of $1+i$ is $\{0\}$:
if $e^{t} \cdot (1+i) = (1+i)$ then $e^{t} = 1$ so $t = 0$

orbit of 0 is $\{0\}$
stabilizer of 0 is all of $\mathbb{R}$

An element $x \in X$ is a fixed point for the action of G if

$$g \cdot x = x \quad \forall g \in G.$$

e.g. the origin in the two actions above.

General Linear Group

Example: $GL_3(\mathbb{R})$ = 3×3 invertible matrizes

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \Bigg| \det A \neq 0$$

group of units in $M_{3\times3}(\mathbb{R})$
(ring of all 3×3 matrices)

acts on $\mathbb{R}^3$: $A \cdot \vec{v} = A\vec{v}$.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax + by + cz \\ dx + ey + fz \\ gx + hy + iz \end{pmatrix}$$

is it an action?
$A \cdot (B \cdot v) = (AB) v$ ✓ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ ✓

orbit of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$: $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ d \\ g \end{pmatrix}$ get any non-zero vector this way.

stabilizer of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$: $\left\{ \begin{pmatrix} 1 & b & c \\ 0 & e & f \\ 0 & h & i \end{pmatrix} \right\} \subset GL_3(\mathbb{R})$

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is a fixed point.

Orbit-Stabilizer theorem

a group G acts on a set X.
for any $x \in X$, $\quad |O_x| \cdot |G_x| = |G|$

Proof: first assume $|O_x|$ and $|G_x|$ are finite

$$|G| = \sum_{y \in O_x} \# \text{ g's in } G \text{ that take } x \text{ to } y.$$

$\# \{ g \in G \mid gx = y \}$ → all the same size as G_x

I claim that for any $y \in O_x$, there is a bijection

$$\boxed{G_x = \{ g \in G \mid gx = x \}}$$

$$\updownarrow \qquad g \mapsto g_0 g \qquad g_0^{-1} g \mapsfrom g$$

$$\boxed{\{ g \in G \mid gx = y \}}$$

choose $g_0 \in G$ such that $g_0 \cdot x = y$

if $gx = x$ then $(g_0 g) \cdot x = g_0(gx) = g_0 x = y$

if $gx = y$ then $(g_0^{-1} g) = g_0^{-1}(gx) = g_0^{-1} y$

$= g_0^{-1}(g_0 x) = (g_0^{-1} g_0) x$

$= 1 \cdot x = x.$

these two maps are inverse bijections:

$$g \mapsto g_0 g \mapsto g_0^{-1} g_0 g = g \qquad\qquad g \mapsto g_0^{-1} g \mapsto g_0 g_0^{-1} g = g$$

$\boxed{\text{QED}}$

Infinite case: less interesting. Think about it.