Last time: Orbit - Stabilizer Theorem.

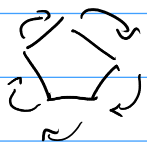
G acts on a set X

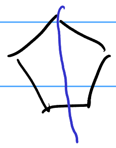
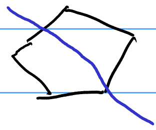
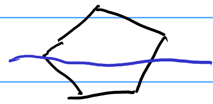
$x \in X$

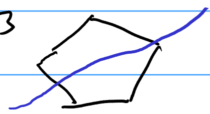
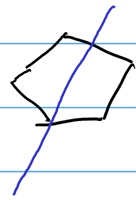
$\rightsquigarrow \ |\mathcal{O}_x| \cdot |G_x| = |G|$ (order of the group = # of elements)

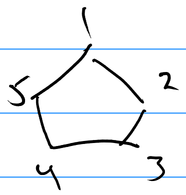
Example to clarify the proof:

$G = D_5$ $\qquad$ X = vertices of a pentagon

$D_5 = \{ 1,\ r$  $,\ r^2,\ r^3,\ r^4,$

s  , sr  sr^2 

sr^3  sr^4 



if x = vertex 1

then $\mathcal{O}_x = \{1, 2, 3, 4, 5\}$

and $G_x = \{1, s\}$

We said that $|G| = \sum_{y \in \mathcal{O}_x} \#\{g \in G \mid gx = y\}$

and for each $y \in \mathcal{O}_x$, that subset of G has the same size as the stabilizer G_x.

$$D_5 = \left\{ \begin{matrix} 1 & r & r^2 & r^3 & r^4 \\ s & sr & sr^2 & sr^3 & sr^4 \end{matrix} \right\}$$

$\{g \in G \mid g \cdot 1 = 1\} = \{1, s\} = H$ $\qquad rs = sr^{-1}$

$\{g \in G \mid g \cdot 1 = 2\} = \{r, \cancel{sr^4}\ rs\} = rH$

$1, s$ take $1 \to 1$

r takes $1 \to 2$

so $r \cdot 1$, $r \cdot s$ both take $1 \to 2$

$\{g \in G \mid g \cdot 1 = 3\} = \{r^2, sr^3 = r^2 s\} = r^2 H$

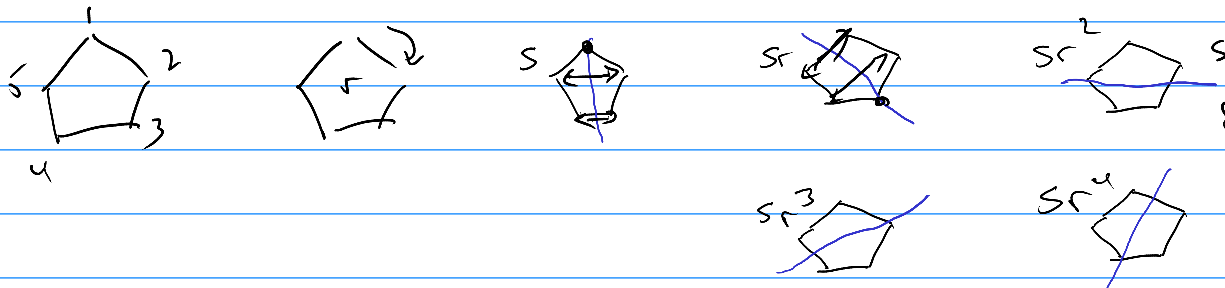
$\{g \in G \mid g \cdot 1 = 4\} = \{r^3, sr^2 = r^3 s\} = r^3 H$

$\{g \in G \mid g \cdot 1 = 5\} = \{r^4, sr = r^4 s\} = r^4 H$

{g's that take 1 to 1} = H $\qquad$ {g's that take 1 to 3} = $r^2 h$

$$h \longmapsto r^2 h$$
$$r^{-2} g \longleftarrow g$$

if $g(1) = 3$ then $r^{-2}(g(1)) = r^{-2}(3) = 1$ so $r^{-2} g = h$ for some $h \in H$



so $g = r^2 h$ for some $h \in H$

Def if G is a group and H is a subgroup and $g \in G$, then the <u>left coset</u>

$$gH = \{gh \mid h \in H\}.$$

or the <u>right coset</u>

$$Hg = \{hg \mid h \in H\}.$$

Just saw that if G acts on a set X and H = stabilizer of some element $x \in X$, then for a given $y \in X$, the set $\{g \in G \mid gx = y\}$ is a left coset of H.

notice: rH = everything that takes vertex 1 to vertex 2

but rs also takes 1 to 2... is rsH different from rH?

no: $rsH = \{rs \cdot 1, rs \cdot s = r\} = rH$.

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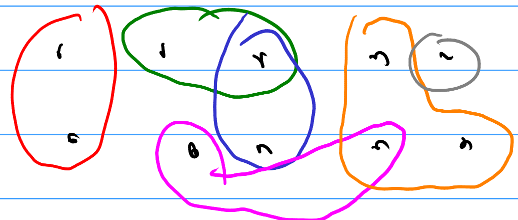
two cosets g_1H and g_2H are either disjoint, or they're equal, so cosets partition the group

Can't have this:



Def: the <u>index</u> of H in G is the # of (left?) cosets, denoted $[G:H]$

Lagrange's thm: $[G:H] = |G|/|H|$ if everything is finite.

$$D_5 = \left\{ \begin{array}{ccccc} H & rH & r^2H & r^3H & r^4H \\ 1 & r & r^2 & r^3 & r^4 \\ s & sr & sr^2 & sr^3 & sr^4 \\ sH & & & & sr^4H \end{array} \right\}$$

$s^2 = 1 \quad r^5 = 1 \quad rs = sr^{-1}$

Difference between left + right cosets?

$$D_5 = \left\{ \begin{array}{ccccc} 1 & r & r^2 & r^3 & r^4 \\ s & sr & sr^2 & sr^3 & sr^4 \end{array} \right\}$$

$$\begin{array}{ccccc} H & Hr & Hr^2 & Hr^3 & Hr^4 \end{array}$$

Another example: $H = \langle r \rangle = \{1, r, r^2, r^3, r^4\} \subset D_5$

$$\left\{ \begin{array}{ccccc} 1 & r & r^2 & r^3 & r^4 \\ s & sr & sr^2 & sr^3 & sr^4 \end{array} \right\} \begin{array}{l} H \\ sH = Hs \end{array}$$

$Hs = \{1 \cdot s, r \cdot s, r^2 \cdot s, r^3 s, r^4 s\}$

$= \{s, sr^4, sr^3, sr^2, sr\}$

Def: a subgroup $H \subset G$ is normal if its left and right cosets are the same:

$gH = Hg \quad \forall g \in G$

Def G/H = set of left cosets of H
$= \{gH \mid g \in G\}$

Thm. (eventually): if H is a normal subgroup then G/H inherits a group structure from G.