G acting on itself...

G acts on itself by left multiplication:

$X = G$, and for $g \in G$ and $x \in G$
we define $g \cdot x = gx$

orbit of 1 under this action?
$g \cdot 1 = g$ ... can get anything this way
stabilizer of 1?
if $g \cdot 1 = 1$ then $g = 1$
stab. is $\{1\}$

More interesting:
G acts on the set of subsets of G:

for $A \subset G$, define $g \cdot A = \{ga \mid a \in A\}$

example: $G = D_4$ $A = \{1, r, s\}$ oo not a subgroup, just a random subset

if $g = r$ then
$gA = rA = \{r, r^2, rs = sr^3\}$

Check: $1 \cdot A = A$ $g \cdot (h \cdot A) = (gh) \cdot A$
so it's a group action.

notice that $|gA| = |A|$
so G acts on the set of 1-element subsets
or 2-element subsets, or 3, ...

especially interesting if $A = H$ is a subgroup

then gH is a left coset of H

orbit of H is the set of all left cosets, aka G/H

size of the orbit = # of cosets

= index of H in G = $[G:H]$

stabilizer? I claim that $gH = H$ iff $g \in H$

so $\operatorname{stab}(H) = H$

orbit-stabilizer thm says

cosets $\cdot |H| = |G|$ (Lagrange!)

example: $G = D_4$

$H = \{\underline{1}, s\}$ $s^2 = 1$, so it's a subgroup.

for $g \in G$, $gH = \{\underline{g, gs}\}$

if $g = 1$ then $1 \cdot H = \{1, s\} = H$

if $g = s$ then $s \cdot H = \{s, 1\} = H$

so $H \subset$ Stabilizer of H

if $gH = H$ then

either $g = 1$ and $gs = s$

or $g = s$ and $gs = 1$

so stabilizer of $H \subset H$.

In general, maybe a HW problem?

Very Interesting:

G acts on itself by conjugation:
for $g \in G$, $x \in G$,

define $g \cdot x = gxg^{-1}$

studied with rotations of a cube on worksheet Fri's

if x spins the front of the cube
and g takes the front to the top
then gxg^{-1} spins the top..

Example: $G = D_5 = \{1, r, r^2, r^3, r^4, s, sr, sr^2, sr^3, sr^4\}$

$rs = sr^{-1}$

$r^k s = sr^{-k}$

Orbit of 1?

$g \cdot 1 \cdot g^{-1} = 1 \quad \forall g.$

orbit $= \{1\}$ $\qquad$ Stab $= G$

Orbit of r? if $g = r^k$ then $grg^{-1} = r^k r r^{-k} = r$

$s^2 = 1$ so $s^{-1} = s$

if $g = sr^k$ then $grg^{-1} = sr^k r r^{-k} s$

$= srs$

$= ssr^{-1} = r^{-1}$

Similarly, $gr^2g^{-1} = r^2$ if g is a rotation
r^{-2} if g is a reflection

Orbit of s? if $g = r^k$ then $gsg^{-1} = r^k s r^{-k}$

all reflections are in 1 orbit.

$= sr^{-k} r^{-k}$

$= sr^{-2k}$