G acting on itself...

G acts on itself by left multiplication:

$X = G$, and for $g \in G$ and $x \in G$
we define $g \cdot x = gx$

orbit of 1 under this action?
g \cdot 1 = g \ldots$ can get anything this way
stabilizer of 1?
if $g \cdot 1 = 1$ then $g = 1$
so $\text{stab}_G$ is $\{1\}$

More interesting:
G acts on the set of subsets of G:

for $A \subset G$, define $g \cdot A = \{ga \mid a \in A\}$

example: $G = D_4$, $A = \{1, r, s\}$

if $g = r$ then
$gA = rA = \{r, r^2, rs = sr^3\}$

check: $1 \cdot A = A$  $g \cdot (h \cdot A) = (gh) \cdot A$
so it's a group action.

notice that $|gA| = |A|$
so G acts on the set of 1-element subsets or 2-element subsets, or 3,...
especially interesting if $A = H$ is a subgroup

then $gH$ is a left coset of $H$
orbit of $H$ is the set of all
left cosets, aka $G/H$
size of the orbit = # of cosets
= index of $H$ in $G$ = $[G : H]$

stabilizer? I claim that $gH = H$
iff $g \in H$
so $\text{stab}(H) = H$
orbit-stabilizer theorem says
# cosets = $[H] = |G|$ (Lagrange!)

example: $G = D_4$
$H = \{1, s\}$
$s^2 = 1$, so it's a subgroup.

for $g \in G$, $gH = \{g, gs\}$
if $g = 1$ then $1H = \{1, s\} = H$
if $g = s$ then $sH = \{s, 1\} = H$
so $H \leq \text{Stabilizer of } H$

if $gH = H$ then
either $g = 1$ and $gs = s$
or $g = s$ and $gs = 1$
so stabilizer of $H \leq H$.

In general, maybe a HW problem?
Very interesting:

$G$ acts on itself by conjugation:

for $g \in G$, $x \in G$,

define $g \cdot x = gxg^{-1}$

studied with rotation of a cube or worksheet Fri's

if $x$ spins the front of the cube and $g$ takes the front to the top,
then $gxg^{-1}$ spins the top.

Example: $G = D_5 = \{1, r, r^2, r^3, r^4, s, sr, sr^2, sr^3, sr^4\}$

Orbit of $1$?

$g \cdot 1 \cdot g^{-1} = 1 \quad \forall_g$

$s \cdot 1 \cdot s^{-1} = s$

$sr \cdot 1 \cdot (sr)^{-1} = sr$

Orbit of $r$?

if $g = r^k$ then $grg^{-1} = r^k \cdot r^{-k} = r$

if $g = sr^k$ then $grg^{-1} = sr^k \cdot r^{-k} = s$

Similarly, $grg^{-1} = r^k$ if $g$ is a rotation

$r^2$ if $g$ is a reflection

Orbit of $s$?

if $g = r^k$ then $gsrg^{-1} = r^k \cdot s \cdot r^{-k}$

all reflections are in 1 orbit.

$7 = s \cdot r^2 \cdot r^{-4}$