Elisa's office hours today are cancelled
Tuesday: 3:10 – 5 as usual
Thursday: me 2 – 3.

We said an action of a group G on a set X is a map

$$G \times X \longrightarrow X$$
$$g, x \longmapsto g \cdot x$$

satisfying
$$g \cdot (h \cdot x) = (gh)\, x \quad \forall g, h \in G \ \forall x \in X$$
$$1 \cdot x = x \quad \forall x \in X$$

Shifrin: An action of G on X is a homomorphism $G \to$ permutations of X.

form a group under composition.

Why is this equivalent?

given a hom $\varphi : G \to$ permutations of X get an action by my def by setting

$$g \cdot x = \varphi(g)\,(x)$$

$\varphi(g)$ is a map from X to X.

then
$$g \cdot (h \cdot x) = \varphi(g)\big(\varphi(h)(x)\big)$$
$$= \big(\varphi(g) \circ \varphi(h)\big)(x) \quad \text{by def of } \circ$$
$$= \varphi(gh)\,(x) \quad \text{because } \varphi \text{ is a hom}$$
$$= (gh) \cdot x$$

also because φ is a hom, $\varphi(1) = 1$
where the first 1 is in G
and the second 1 is the identity map $X \to X$

so $1 \cdot x = \varphi(1)(x) =$ identity map $(x) = x$.

go the other way: given an action $G \times X \longrightarrow X$

define $\varphi: G \longrightarrow$ permutations of X
$g \longmapsto (x \longmapsto g \cdot x)$

check: because of the two axioms for an action, φ is a homomorphism.
really: also check that $\forall g \in G$, the map $X \longrightarrow X$ is a bijection.
$x \longmapsto g \cdot x$

Fun observation: every group is iso. to a subgroup of a permutation group.

Seen: G acts on itself by left mult.
For $g \in G$ and $x \in G$, defined $g \cdot x = gx$

so we get a hom.
$G \longrightarrow$ permutations of the set G
$g \longmapsto (x \longmapsto gx)$

it's injective! if $g \longmapsto$ identity
then $gx = x \ \forall x \in G$ so $g = 1$.

Example: let $G = D_4$. $|G| = 8$

so we get an inj. hom.

$$G \hookrightarrow S_8$$

size $= 8$ $\qquad$ size $= 8! = 40320$

Actions of $\mathbb{Z}_2$

$\mathbb{Z}_2 = \{0, 1\}$ under addition

$\cong \{1, -1\}$ under multiplication

An action of $\mathbb{Z}_2$ on a set X is the same as a map $i : X \longrightarrow X$

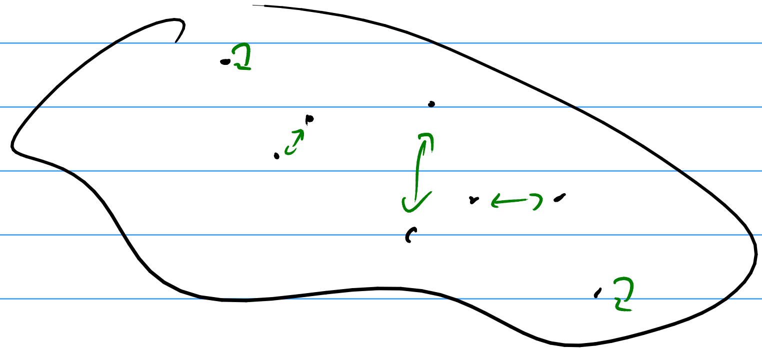
s.t. $i \circ i = 1$

$i(i(x)) = x \quad \forall x \in X$

i is an "involution"

define $i(x) = -1 \cdot x$ using $\mathbb{Z}_2 \cong \{1, -1\}$

X



notice: if $\mathbb{Z}_2$ acts on a set X with $|X| = $ odd then there is a fixed point.

Shifrin's notation $\# \mathcal{O}_x$ $x \in X$
is the same as my $|\mathcal{O}_x|$.
Just the number of elements in $\mathcal{O}_x$.

Seen: if a group G acts on a set X
and I choose some $x \in X$,
let $\mathcal{O}_x$ = orbit of x $\subset X$
G_x = stabilizer of x $\subset G$

left cosets of G_x $\longleftrightarrow$ elements $y \in \mathcal{O}_x$

$gG_x \longleftrightarrow y = g \cdot x$

(review last Monday's notes...)

but different elements of $\mathcal{O}_x$ may have different stabilizers.

How is G_x related to G_y if $\underline{y = gx}$?

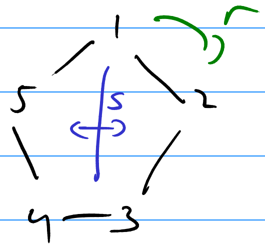
Same size for sure...
but we can say more.

$$
\begin{aligned}
a \in G_x &\iff ax = x \\
&\iff a \cdot g^{-1} y = g^{-1} y \\
&\iff \\
&\iff g a g^{-1} y = y \\
&\iff g a g^{-1} \in G_y .
\end{aligned}
$$

So $\quad G_y = gG_xg^{-1} = \{ gag^{-1} \mid a \in G_x \}$

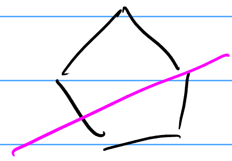
if you like, $\quad G_{gx} = gG_xg^{-1}$.

Example: $\quad G = D_5 \quad X =$ vertices of a pentagon.



stabilizer of 1 is $\{1, s\}$

stabilizer of 2 is $\{1, \underbrace{\text{some other reflection}}_{sr^{-2} = sr^{3}}\}$



$2 = r \cdot 1$

$1 = r^{-1} \cdot 2$

$s \cdot 1 = 1 \qquad sr^{-1} \cdot 2 = 1 \qquad rsr^{-1} \cdot 2 = 2$

$$rsr^{-1} = sr^{-1}r^{-1} = sr^{-2}$$

Quotient Groups

For rings: if $\varphi: R \to S$ is a homomorphism
then $\ker \varphi = \{ r \in R \mid \varphi(r) = 0 \}$
is an ideal,
$\operatorname{im} \varphi \subset S$ is a subgroup,
and we get an iso $R/\ker\varphi \longrightarrow \operatorname{im}\varphi$

For groups: if $\varphi: G \to H$ is a homomorphism
then $\ker \varphi = \{ g \in G \mid \varphi(g) = 1 \}$
is a <u>normal subgroup</u>,

$\operatorname{im} \varphi \subset H$ is a subgroup,

and we get an iso $G/\ker\varphi \longrightarrow \operatorname{im}\varphi$.

Comments

1. A subgroup $N \subset G$ is called <u>normal</u> if

$\forall g \in G \ \forall h \in N$ we have $ghg^{-1} \in N$.

don't require $ghg^{-1} = h$!

if G is Abelian then every subgroup is normal
bec. $ghg^{-1} = gg^{-1}h = h$.

$\ker \varphi$ is normal (for any hom. $\varphi: G \to H$)
because if $g \in G$ and $k \in \ker \varphi$

then $\varphi(gkg^{-1}) = \varphi(g)\varphi(k)\varphi(g)^{-1}$ bec. φ is a hom

$= \cancel{\varphi(g)} \cdot 1 \cdot \cancel{\varphi(g)^{-1}}$ bec. $k \in \ker \varphi$

$= 1$

So $gkg^{-1} \in \ker \varphi$.

Example: in D_5, $\{1, s\}$ is not a normal subgroup, because

$rsr^{-1} = sr^3$ is not in there.

But $\langle r \rangle = \{1, r, r^2, r^3, r^4\}$ is normal

because $(sr^m) r^n (sr^m)^{-1} = r^{-n}$
and $(r^m) r^n (r^m)^{-1} = r^n$
both are in $\langle r \rangle$.

Can we see $\langle r \rangle$ as the kernel of some hom. $\varphi: D_5 \longrightarrow$ somewhere?

Yes: define $\varphi: D_5 \longrightarrow \mathbb{Z}_2 = \{1, -1\}$ under mult.

by $\varphi(g) = +1$ if g is a rotation
$\varphi(g) = -1$ if g is a reflection.

check: it's a hom.

(2) If N is normal then
G/N = set of left (or right) cosets of N

inherits a group structure: $g_1N \cdot g_2N = (g_1g_2)N$ is well-defined.