Friday's worksheet:
wrote some cycles as products of transpositions

Recall the symmetric group or permutation group

$S_n = \{$ bijections from $\{1, 2, \dots, n\}$ to itself $\}$
operation is composition

$|S_n| = n!$

a <u>transposition</u> is an element $\tau \in S_n$ that switches 2 elts of $\{1, 2, \dots, n\}$ and leaves the rest alone

notice: a transposition is a 2-cycle

a <u>cycle</u> is an element $\sigma \in S_n$ that takes some $i_1 \in \{1, 2, \dots, n\}$ to i_2, i_2 to i_3, ... eventually back to i_1 and leaves the rest alone.

examples: $\sigma = 1 \to 2 \to 3 \to 4$ in S_4 is a 4-cycle

$\sigma' = 1 \to 3 \to 2 \to 4$ in S_4 is also a 4-cycle

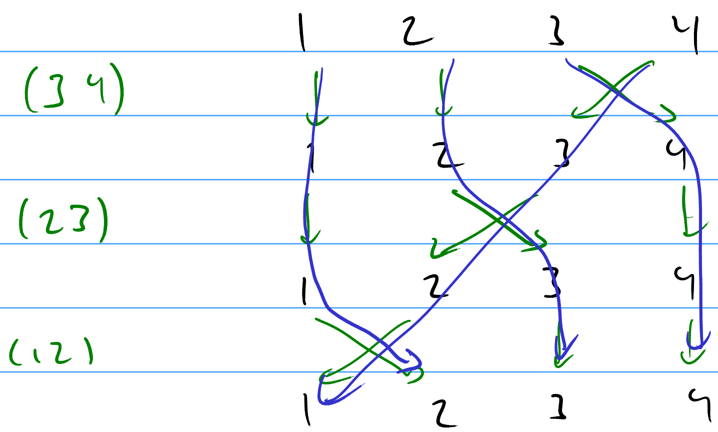
new notation, called <u>cycle notation</u>:

$\sigma = (1\ 2\ 3\ 4)$

$\sigma' = (1\ 3\ 2\ 4)$

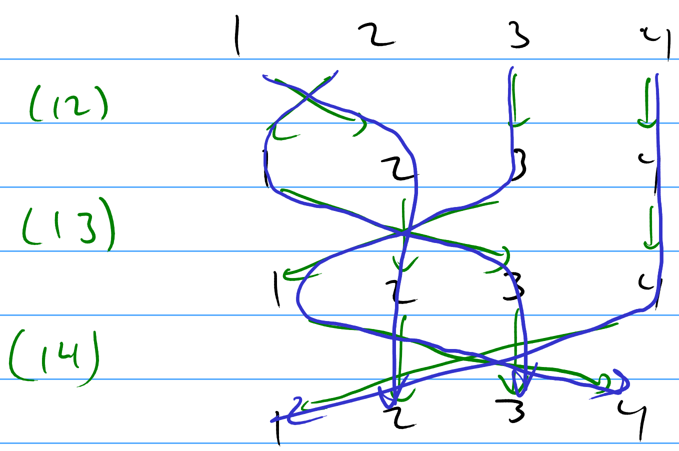
could also have called σ $(2\ 3\ 4\ 1)$
but convention is to put the smallest # first.

you found that $(1\ 2\ 3\ 4) = (1\ 2)(2\ 3)(3\ 4)$



could also have done

$$(1\,2\,3\,4) = (1\,4)(1\,3)(1\,2) \leftarrow$$



You also found that
$(1\ 3\ 2\ 4) = (2\ 3)(1\ 2)(2\ 3)(3\ 4)(2\ 3)$ 5

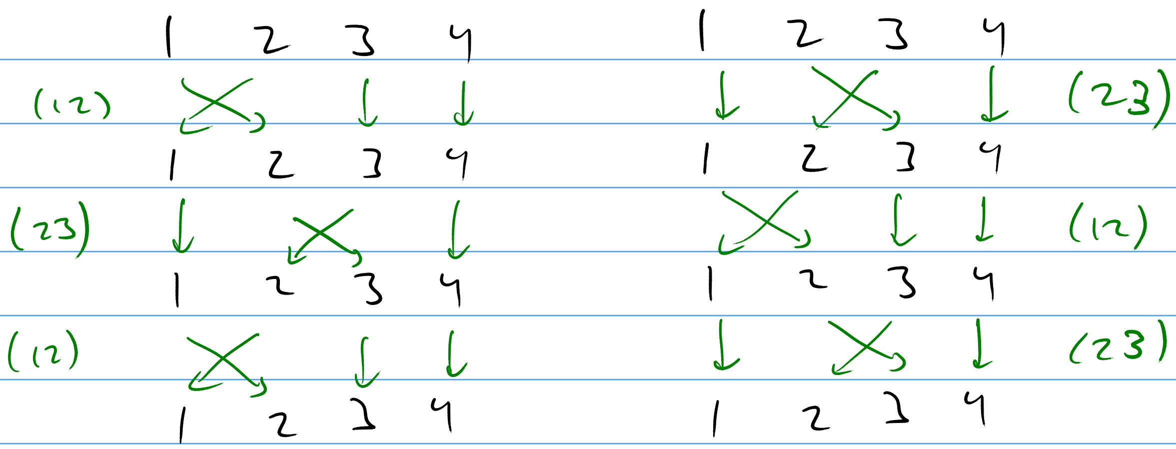
also true:
$(1\ 3\ 2\ 4) = (1\ 3)(3\ 2)(2\ 4)$ 3

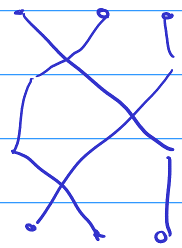
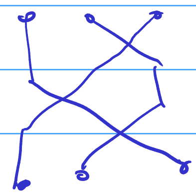
or $(1\ 3\ 2\ 4) = (1\ 4)(1\ 2)(1\ 3)$ another 3

Prop: every cycle can be written as a product of transpositions. but not uniquely!

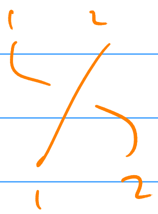
Proof: use the idea above. or read in §6.4 $\square$

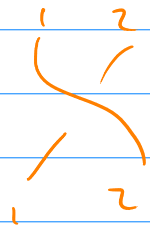
also saw that $(12)(23)(12) = (23)(12)(23) = (13)$



 =  "braid relation"

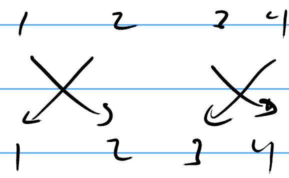
aside: $\exists$ a thing called the braid group

where 

is not the same as 

...

also saw that $(12)(34) = (34)(12)$



notice: $(12)(34)$ is a permutation that's not a cycle. rather, it's a product of 2 disjoint cycles.

<u>Prop</u>: every element of S_n can be written as a product of disjoint cycles.

uniquely up to reordering the cycles.

So every element of S_n can be written as a product of transpositions.

<u>Parity</u> or sign of a permutation:

the map $S_n \longrightarrow \mathbb{Z}_2$

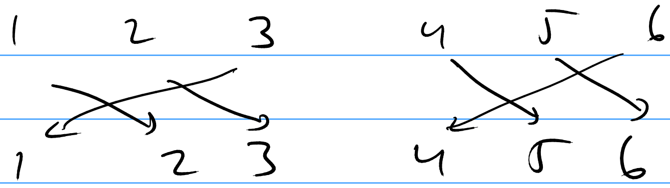
$\sigma \longmapsto$ how many transpositions you need to multiply to get σ, mod 2

<u>Thm</u>: it's well-defined! if σ is a product of an odd # of transpositions, then it can't be written as a product of an even # of transpositions.

Also this is a homomorphism.

not obvious! proof next time

Example: in S_6, take $\sigma = (1\,2\,3)(4\,5\,6)$



$$\sigma = (1\,2)(2\,3)\,(4\,5)(5\,6)$$

so σ is an <u>even</u> permutation.