Sign of a Permutation

Last time: cycle notation.
every permutation is a product of disjoint cycles (uniquely)
every cycle is a prod. of transpositions (not uniquely)

Sign: $S_n \longrightarrow \mathbb{Z}_2$

$\sigma \longmapsto$ write $\sigma = \tau_1 \circ \tau_2 \circ \dots \circ \tau_k$ (prod of transpositions)
and take $k \pmod 2$

well-defined??
if it is then clearly a hom.

Example: an n-cycle is an even perm. iff n is odd.
$(1342) = (13)(34)(42)$
4-cycle = prod. of 3 transpositions

Remark: Could define $\operatorname{sign}(\sigma)$ to be
det of the matrix obtained from

$\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix}$ by permuting the cols. according to σ.

sign of $(1\,3\,4\,2)$

$$= \det \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = -1$$

but actually the def of determinant
requires the sign of a permutation...

if $A = (a_{ij})$ an $n \times n$ matrix

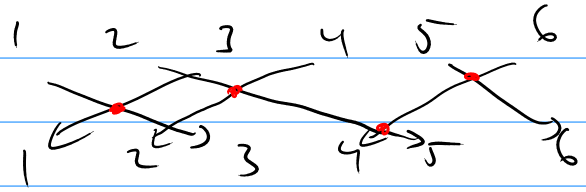
then $\det A = \sum_{\sigma \in S_n} (-1)^{\text{sign}(\sigma)} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$

Another def: if $\sigma \in S_n$,

define $\text{sign}(\sigma) = \#\{\text{pairs } i,j \in \{1,\dots,n\} : i<j \text{ but } \sigma(i) > \sigma(j)\} \bmod 2.$

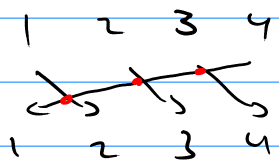
Counts # of crossings in that diagram

Example: from worksheet, $\underbrace{(1\,3)}_{\text{odd}}\underbrace{(2\,5\,6\,4)}_{\text{odd}}$ should be even

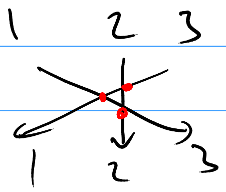


$\#\{(1,3)\ (2,4)\ (2,6)\ (5,6)\} \bmod 2 = 4 \bmod 2$
$= 0$

another one: $(1\,2\,3\,4)$ is odd



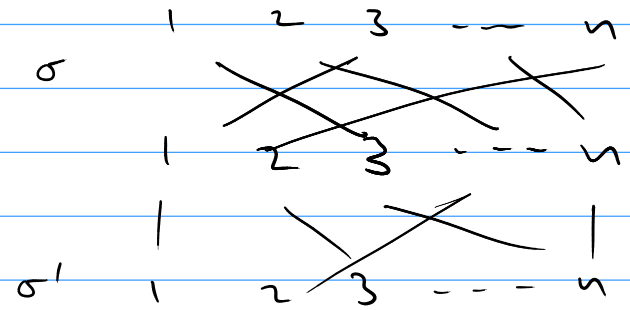
or $(1\,3)$ is odd



This is clearly well-defined but not clear that it's a homomorphism...

Want: if $\sigma, \sigma' \in S_n$ then
$\mathrm{sign}(\sigma' \circ \sigma) = \mathrm{sign}(\sigma') + \mathrm{sign}(\sigma)$ in $\mathbb{Z}_2$
(with new def. of sign)

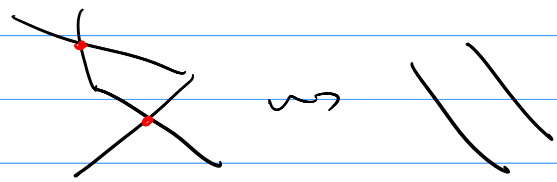
Intuitively:



← maybe this has k crossings.

← ℓ crossings.

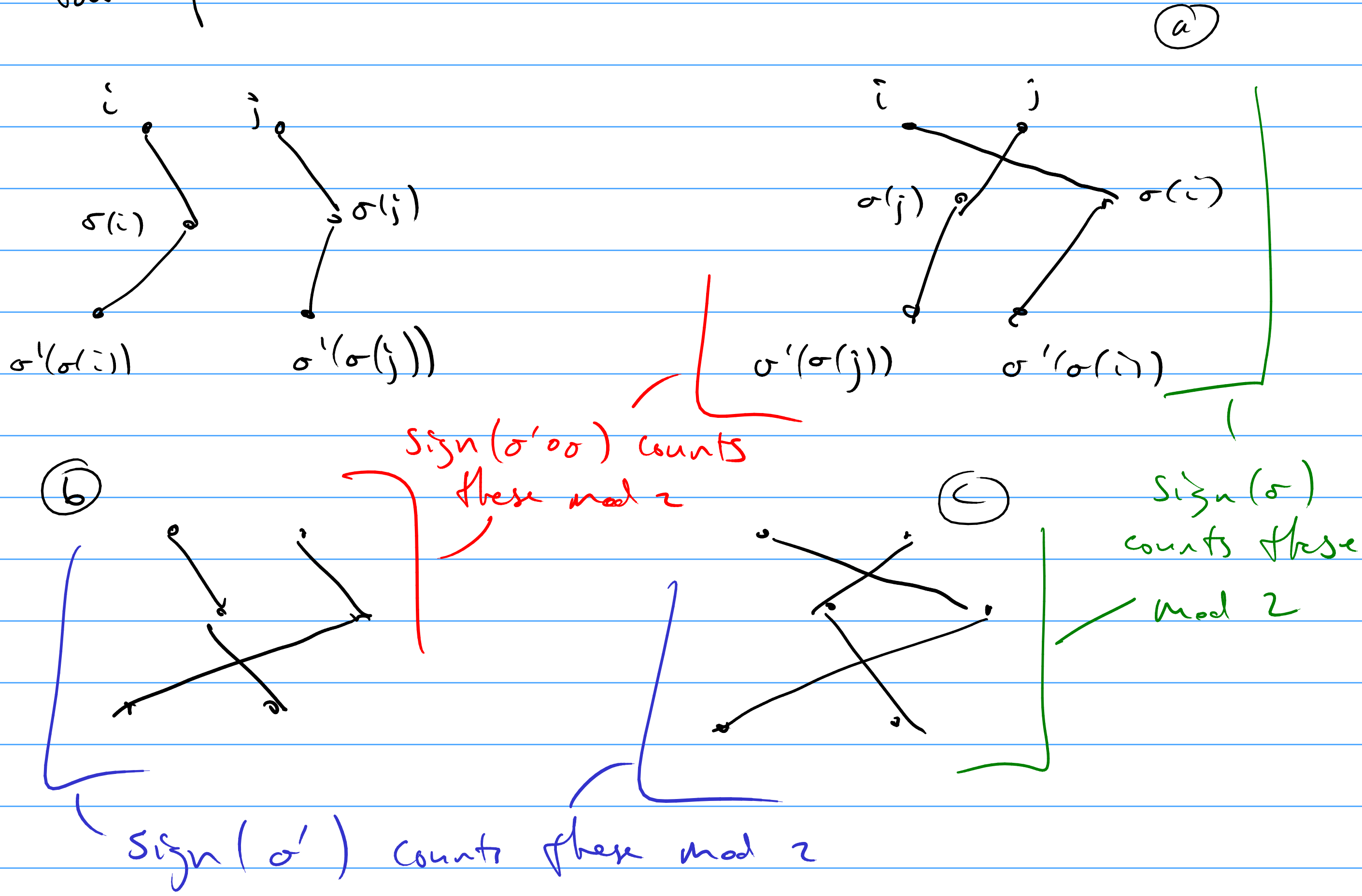
to find $\sigma' \circ \sigma$, stack them up ($k+\ell$ crossings) and straighten out all the bent paths.
↳ might reduce # of crossings but only by an even number.



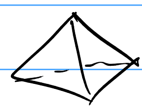
So $\mathrm{sign}(\sigma) + \mathrm{sign}(\sigma') = \mathrm{sign}(\sigma' \circ \sigma)$ in $\mathbb{Z}_2$

A little more formal:
given $i, j \in \{1, 2, \dots, n\}$
with $i < j$,
ask whether $\sigma(i) < \sigma(j)$ or $>$
and whether $\sigma'(\sigma(i)) < \sigma'(\sigma(j))$ or $>$

four possibilities:



and $a+c + b+c = a+b+2c \equiv a+b \pmod 2$.

Worksheet: symmetries of a tetrahedron 

vs S_4. even permutations $\longleftrightarrow$ rotations.