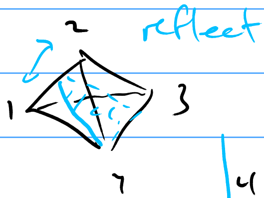
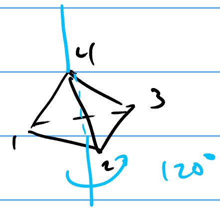
Been talking about even + odd permutations.

In S_4, you've got

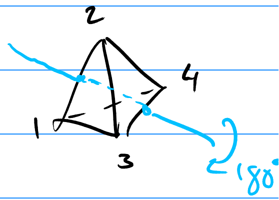
- 1
- transpositions (1 2) etc →
  ↳ 6 of these



- 3-cycles (1 2 3) (1 3 2) etc
  ↳ 8 of these



- 4-cycles (1 2 3 4) (1 3 2 4) etc hard to visualize!
  ↳ 6 of these
- products of disjoint 2-cycles (12)(34) etc.
  ↳ 3 of these



(odd: transpositions, 4-cycles; even: 1, 3-cycles, products of disjoint 2-cycles)

total = 24 = 12 even ones + 12 odd ones

Worksheet: symmetries of a tetrahedron ≅ S_4

rotations ⟷ even perms.

Classify Groups of Order ≤ 7

Theorem: Let G be a group.

- if $|G| = 2$ then $G \cong \mathbb{Z}_2$ ✓
- if $|G| = 3$ then $G \cong \mathbb{Z}_3$ ✓
- if $|G| = 4$ then $G \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ ←
- if $|G| = 5$ then $G \cong \mathbb{Z}_5$ ✓
- if $|G| = 6$ then $G \cong \mathbb{Z}_6$ or S_3 ←
- if $|G| = 7$ then $G \cong \mathbb{Z}_7$ ✓

($|G| = 8$ becomes more interesting...)

Main Tool: Lagrange's thm:
order of a subgroup divides $|G|$
order of an element divides $|G|$

Lemma 1: if $|G| = k$ and G has an element g of order k then $G \cong \mathbb{Z}_k$.

Proof: look at the homomorphism

$$\mathbb{Z}_k \to G$$
$$\bar{n} \longmapsto g^n$$

$$\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{k-1} \longmapsto 1, g, g^2, \ldots, g^{k-1}$$

it's an isomorphism.

If $|G|$ is prime then every elt other than 1 has order p so $G \cong \mathbb{Z}_p$.

Lemma 2: if every elt. of G has order 1 or 2 then G is Abelian.

Proof: let $a, b \in G$. then $a^2 = 1$ and $b^2 = 1$

and $(ab)^2 = 1$

so $abab \cdot b = 1 \cdot b$

$aba \cdot a = b \cdot a$

$ab = ba$ ☒

Now suppose $|G| = 4$.

every elt must have order 1, 2, or 4

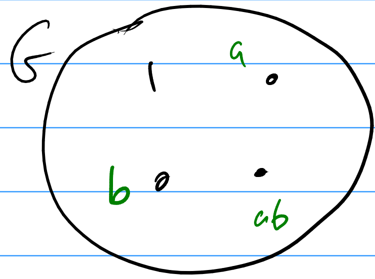
and only one has order 1.

- if one elt has order 4, then $G \cong \mathbb{Z}_4$
- if all have order 1 or 2, claim that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$...

let $a, b \in G$ $a \neq 1$ $b \neq 1$ $a \neq b$.

assuming $a^2 = 1$ $b^2 = 1$ $ab = ba$ by lemma 2

then ab is the fourth elt:



$ab \neq a$ bec. $b \neq 1$

$ab \neq b$ bec. $a \neq 1$

$ab \neq 1$ bec $b \neq a = a^{-1}$

Then the map $\mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow G$

$(0,0) \longmapsto 1$

$(1,0) \longmapsto a$

$(0,1) \longmapsto b$

$(1,1) \longmapsto ab$

is an iso.

☒

Last, suppose $|G| = 6$.
then every elt. has order 1, 2, 3, or 6.

if there's an elt. of order 6 then $G \cong \mathbb{Z}_6$.

if all order 2 $\rightsquigarrow$ G is Abelian
+ contains a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, as before
but $|\mathbb{Z}_2 \times \mathbb{Z}_2| = 4$ which doesn't divide 6.

if all order 3 $\rightsquigarrow$ non-identity elts.
come in pairs, a with a^{-1}
so $|G|$ would have to be odd!

So let $a \in G$ have order 3
and b have order 2.

get $1, a, a^2$ all distinct
b, ab, a^2b all distinct

if $ba = ab$ then we'll see that
$G \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$

if $ba = ab^2$ then we'll see that
$G \cong D_3 \cong S_6$

more detail next time...