Quotient Groups

G = a group
$H < G$ a subgroup.

Def: H is normal if
$\forall g \in G \ \forall h \in H$ we have $ghg^{-1} = H$

Recent HW: equivalent to say
$\forall g \in G$ we have $gH = Hg$.

Saw an example with D_5...
$H = \langle s \rangle$ is not normal... for example, rsr^{-1}
but $H = \langle r \rangle$ is normal. $= sr^{-2}$
$= sr^{3} \notin \langle s \rangle$
$(srs^{-1} = srs = ssr^{-1} = r^{-1} = r^{4}$

Def: G/H = set of all (left) cosets gH

On recent HW, you showed that
$aH = bH$ iff $b^{-1}a \in H$
equivalently: $b^{-1}a = h$ for some $h \in H$
iff $a = bh$ for some $h \in H$

Thm: if $N < G$ is a normal subgroup
then the operation
$$aN \cdot bN = abN$$
is well-defined and makes G/N into a group.

Pf: Well-defined means if $aN = a'N$ and $bN = b'N$ then $abN = a'b'N$

if $a'N = aN$ then $a' = an_1$ for some $n_1 \in N$
if $b'N = bN$ then $b' = bn_2$ for some $n_2 \in N$

then $a'b' = an_1 bn_2$

$$= ab \cdot b^{-1} n_1 b n_2$$

$$= ab \cdot \underbrace{\underbrace{(b^{-1} n_1 b)}_{\in N \text{ because } N \text{ is normal}} \cdot n_2}_{\text{so all of this is in } N.}$$

So $abN = a'b'N$.

Now is G/N a group under this operation?

Associative: $(aN \cdot bN) \cdot cN = abN \cdot cN$
$= abcN$
$= aN \cdot bcN$
$= aN \cdot (bN \cdot cN)$

Identity is N a.k.a. $1N$

$aN \cdot 1N = (a \cdot 1)N = aN$

similarly $1N \cdot aN = aN$

Inverses: $aN \cdot a^{-1}N = aa^{-1}N = 1N$

similarly $a^{-1}N \cdot aN = 1N$. ☐

Understand quot groups using the 1st iso thm:

if $\varphi: G \to H$ is a hom.

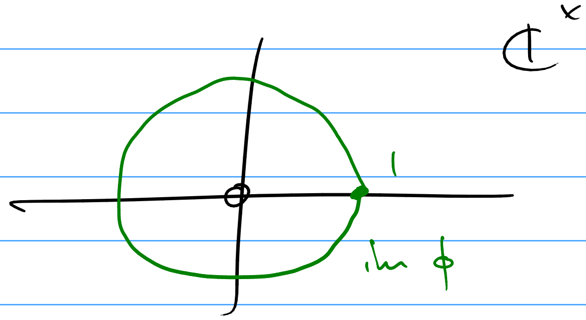
then $\underbrace{G/\ker\varphi}_{\text{normal subgroup of } G} \cong \underbrace{\operatorname{im}\varphi}_{\text{subgroup of } H.}$

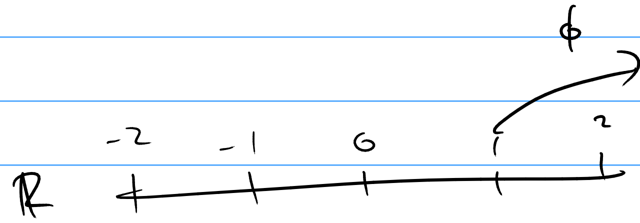
Pf: next time.

Example from HW8:

$$\phi: \mathbb{R} \longrightarrow \mathbb{C}^\times$$
$$t \longmapsto \cos 2\pi t + i \sin 2\pi t = e^{2\pi i t}$$

$\operatorname{im}\phi$ = unit circle





$$\ker\phi = \{t \in \mathbb{R} \mid \phi(t) = 1\} = \mathbb{Z} \subset \mathbb{R}$$

Thm: $\mathbb{R}/\mathbb{Z} \cong$ unit circle as a subgroup of $\mathbb{C}^\times$

cosets of $\mathbb{Z}$?

things like $0 + \mathbb{Z}$

$\frac{1}{2} + \mathbb{Z}$

$\frac{1}{3} + \mathbb{Z}$