Homework 1

Due Friday, January 8, 2021

Let R be a commutative ring.

1. Let $a, b \in R$, and let $I \subset R$ be an ideal. Prove that $\langle a \rangle \subset I$ if and only if $a \in I$, and similarly that $\langle a, b \rangle \subset I$ if and only if $a \in I$ and $b \in I$.

2. Let $a \in R$. Prove that a is a unit if and only if $\langle a \rangle = \langle 1 \rangle$.

3. Let $a, b \in R$. Prove that $a \mid b$ if and only if $\langle b \rangle \subset \langle a \rangle$.

4. Let $a, b \in R$. Prove that if there is a unit $u \in R$ such that $b = ua$, then $\langle a \rangle = \langle b \rangle$. Prove that if R is an integral domain then the converse holds as well.

   (For example, in $\mathbb{Z}$ we have $\langle 2 \rangle = \langle -2 \rangle$,
   and in $\mathbb{Q}[x]$ we have $\langle 2x + 1 \rangle = \langle x + \frac{1}{2} \rangle$.)

5. In $\mathbb{Z}[\sqrt{-5}]$, let

$$\begin{aligned} I_1 &= \langle 2, 1 + \sqrt{-5} \rangle \\ I_2 &= \langle 2, 1 - \sqrt{-5} \rangle \\ I_3 &= \langle 1 + \sqrt{-5}, 1 - \sqrt{-5} \rangle. \end{aligned}$$

   Prove that $I_1 = I_2 = I_3$.

   Hint: Use problem 1 to prove that each one is contained in the others. The slickest thing would be to prove that $I_1 \subset I_2 \subset I_3 \subset I_1$.