Solutions to Homework 1

Let R be a commutative ring.

1. Let $a, b \in R$, and let $I \subset R$ be an ideal. Prove that $\langle a \rangle \subset I$ if and only if $a \in I$, and similarly that $\langle a, b \rangle \subset I$ if and only if $a \in I$ and $b \in I$.

   Solution: For the first statement, recall that

   $$\langle a \rangle = \{ra : r \in R\}.$$

   Taking $r = 1$, we see that $a \in \langle a \rangle$, so if $\langle a \rangle \subset I$ then $a \in I$. Conversely, if $a \in I$, then because I is an ideal we have $ra \in I$ for all $r \in R$, so $\langle a \rangle \subset I$.

   For the second statement is similar, recall that

   $$\langle a, b \rangle = \{ra + sb : r, s \in R\}.$$

   Taking $r = 1$ and $s = 0$, we see that $a \in \langle a, b \rangle$, and taking $r = 0$ and $s = 1$ we see that $b \in \langle a, b \rangle$, so if $\langle a, b \rangle \subset I$ then $a \in I$ and $b \in I$. Conversely, if $a \in I$ and $b \in I$, then because I is an ideal we have $ra \in I$ and $sb \in I$ for all $r, s \in R$, and thus $ra + sb \in I$, so we conclude that $\langle a, b \rangle \subset I$.

2. Let $a \in R$. Prove that a is a unit if and only if $\langle a \rangle = \langle 1 \rangle$.

   Solution: By definition, a is a unit if and only if there is an $r \in R$ such that $1 = ra$, which is true if and only if $1 \in \langle a \rangle$, which is true if and only if $\langle 1 \rangle \subset \langle a \rangle$ by problem 1. But we always have $\langle a \rangle \subset \langle 1 \rangle$, because $\langle 1 \rangle$ is the whole ring R, so $\langle 1 \rangle \subset \langle a \rangle$ if and only if $\langle a \rangle = \langle 1 \rangle$.

3. Let $a, b \in R$. Prove that $a \mid b$ if and only if $\langle b \rangle \subset \langle a \rangle$.

   Solution: By definition, $a \mid b$ if and only if there is an $r \in R$ such that $b = ra$, which is true if and only if $b \in \langle a \rangle$, which is true if and only if $\langle b \rangle \subset \langle a \rangle$ by problem 1.

4. Let $a, b \in R$. Prove that if there is a unit $u \in R$ such that $b = ua$, then $\langle a \rangle = \langle b \rangle$. Prove that if R is an integral domain then the converse holds as well.

   Solution: Because $b = ua$ we have $b \in \langle a \rangle$, so $\langle b \rangle \subset \langle a \rangle$ by problem 1. Because u is a unit, we can write $uv = 1$ for some $v \in R$; then we see that $a = vb$, so $a \in \langle b \rangle$, so $\langle a \rangle \subset \langle b \rangle$ as well. Thus $\langle a \rangle = \langle b \rangle$.

   For the converse, if $\langle a \rangle = \langle b \rangle$ then we can write $b = ua$ and $a = vb$ for some $u, v \in R$. Then $a = uva$, so $0 = (uv - 1)a$, and because R is an integral domain, this implies that either $uv - 1 = 0$ or $a = 0$. In the first case, $uv = 1$, so u is a unit. In the second case, $\langle a \rangle = \{0\}$, so $b = 0$ as well, and then we can write $b = 1 \cdot a$.

5. In $\mathbb{Z}[\sqrt{-5}]$, let

$$\begin{aligned} I_1 &= \langle 2, 1 + \sqrt{-5} \rangle \\ I_2 &= \langle 2, 1 - \sqrt{-5} \rangle \\ I_3 &= \langle 1 + \sqrt{-5}, 1 - \sqrt{-5} \rangle. \end{aligned}$$

   Prove that $I_1 = I_2 = I_3$.

   Hint: Use problem 1 to prove that each one is contained in the others. The slickest thing would be to prove that $I_1 \subset I_2 \subset I_3 \subset I_1$.

   Solution: Taking the hint, we observe that $2 \in I_2$, and

$$1 + \sqrt{-5} = 2 - (1 - \sqrt{-5}) \in I_2,$$

   so $I_1 \subset I_2$ by problem 1.

   Next, we have $1 - \sqrt{-5} \in I_3$, and

$$2 = (1 + \sqrt{-5}) + (1 - \sqrt{-5}) \in I_3$$

   so $I_2 \subset I_3$ by problem 1.

   Finally, we have $1 + \sqrt{-5} \in I_1$, and

$$1 - \sqrt{-5} = 2 - (1 + \sqrt{-5}) \in I_1,$$

   so $I_3 \subset I_1$ by problem 1.

   Thus $I_1 = I_2 = I_3$.