Homework 4

Due Friday, January 29, 2021

1. (a) Let $\phi\colon R \to S$ be an isomorphism. Prove that $r \in R$ is a zero-divisor if and only if $\phi(r)$ is a zero-divisor in S.
   (b) Give an example to show that the conclusion of part (a) may fail if ϕ is only a homomorphism. Indicate where your proof from part (a) breaks down if ϕ is not a bijection.
   (c) Let $\phi\colon R \to S$ be an isomorphism. Prove that $r \in R$ is a unit if and only if $\phi(r)$ is a unit in S.
   (d) Give an example to show that the conclusion of part (c) may fail if ϕ is only a homomorphism. Indicate where your proof from part (c) breaks down if ϕ is not a bijection.

2. (a) Prove that $\mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.
   Hint: One approach is to try to prove that they *are* isomorphic, fail a few times, then try to understand why there can't possibly be an isomorphism. After doing that scratch work, your proof should begin "Suppose that $\phi\colon \mathbb{Z}_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ were an isomorphism. Then…" and should end with a contradiction.
   (b) Prove that $\mathbb{Z}_2[x]/\langle x^2 \rangle$ is not isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.
   (c) Prove that $\mathbb{Z}_2[y]/\langle y^2 + y + 1 \rangle$ is not isomorphic to any of the three rings above.
   (d) Prove that $\mathbb{Z}_2[z]/\langle z^2 + z \rangle$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.
   Hint: Cook up a suitable homomorphism $\phi\colon \mathbb{Z}_2[z] \to \mathbb{Z}_2 \times \mathbb{Z}_2$, and apply the first isomorphism theorem. You can reuse some ideas from problem 1(c) on last week's homework.
   (e) Optional: Prove that $\mathbb{Z}_2[w]/\langle w^2 + 1 \rangle$ is isomorphic to $\mathbb{Z}_2[x]/\langle x^2 \rangle$.

3. Let R be the set of 2×2 matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$,
   (a) Show that R is a subring of $M_2(\mathbb{R})$, and that it is commutative.
   (b) Prove that $R \cong \mathbb{R}[x]/\langle x^2 \rangle$.
   Hint: Show that the map $\phi\colon \mathbb{R}[x] \to R$ given by

$$\phi(a + bx + cx^2 + \cdots) = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

   is a homomorphism, determine its kernel, and quote the first isomorphism theorem.