Homework 5

Due Friday, February 5, 2021

1. Recall that an ideal $P \subsetneq R$ is called *prime* if for all $a, b \in R$ with $ab \in P$ we have $a \in P$ or $b \in P$.

a. Suppose that $S \subset R$ is a subring, and let $Q = P \cap S$. Prove that if P is a prime ideal in R then Q is a prime ideal in S.

b. Let $P = \langle 2+i \rangle \subset \mathbb{Z}[i]$, which is a prime ideal because $2+i$ is irreducible. Consider the subring $\mathbb{Z} \subset \mathbb{Z}[i]$ and the intersection $Q = P \cap \mathbb{Z}$. Then Q is a prime ideal of $\mathbb{Z}$, so it's either $\langle 0 \rangle$ or $\langle p \rangle$ for some prime number $p \in \mathbb{Z}$. Which one is it, and if it's the latter, what is p? Give a proof.

c. Let $z \in \mathbb{Z}[i]$ be irreducible. Prove that there is a prime number $p \in \mathbb{Z}$ such that z divides p in $\mathbb{Z}[i]$, by considering the ideal $P = \langle z \rangle$ and the intersection $Q = P \cap \mathbb{Z}$. Conclude that either $|z|^2 = p$ or $|z|^2 = p^2$.

§4.3 #6. Prove that $\mathbb{Z}[i]$ is a principal ideal domain: that is, for every ideal $I \subset \mathbb{Z}[i]$ there is an element $z \in \mathbb{Z}[i]$ such that $I = \langle z \rangle$. Emulate the proof of §4.1 Proposition 1.2, which uses the same ideas as §1.2 Theorem 2.3 and §3.1 Theorem 1.6.

Based on §4.3 #16. Let R be a commutative ring, let $a, b \in R$, let $S = R/\langle a \rangle$, and let $T = S/\langle \bar{b} \rangle$, where $\bar{b} \in S$ is the equivalence class of b modulo $\langle a \rangle$. Prove that $T \cong R/\langle a, b \rangle$.

Hint: Find a surjective homomorphism $R \to T$, prove that its kernel is $\langle a, b \rangle$, and quote the first isomorphism theorem.