Solutions to Homework 6

§6.1 #1. *Which of the following are groups?*

a. $\{1, 3, 7, 9\} \subset \mathbb{Z}_{10}$ *with operation multiplication.*

Solution: Yes. These are the units in the ring $\mathbb{Z}_{10}$, which form a group under multiplication by Proposition 1.2.

b. $\{0, 2, 4, 6\} \subset \mathbb{Z}_{10}$ *with operation addition.*

Solution: No. Addition is not a binary operation on this set, because $2 + 6 = 8$ is not in the set.

c. $\{x \in \mathbb{Q} : 0 < x \leq 1\}$ *with operation multiplication.*

Solution: No. Multiplication on this set is associative, and the identity element is 1, but not every element has an inverse: for example, $\frac{1}{2}$ is in the set, but there is no x in the set such that $x \cdot \frac{1}{2} = 1$.

d. *The set of all positive irrational real numbers, withwith operation multiplication.*

Solution: No. Multiplication is not a binary operation on this set: $\sqrt{2}$ is in the set, but $\sqrt{2} \cdot \sqrt{2} = 2$ is not.

e. *The set of imaginary numbers ix, $x \in \mathbb{R}$, with operation addition.*

Solution: Yes. Addition takes this set into itself: $ix + iy = i(x + y)$. It is associative; the identity is 0; and the additive inverse of ix is $-ix = i(-x)$.

In fact this group is isomorphic to the additive group of $\mathbb{R}$, via the map that takes $x \in \mathbb{R}$ to ix.

f. *The set of complex numbers of modulus 1, with operation multiplication.*

Solution: Yes. Multiplication takes this set into itself: if $|z| = 1$ and $|w| = 1$ then $|zw| = |z||w| = 1$. It is associative; the identity is 1; and if $|z| = 1$ then $|z^{-1}| = 1/|z| = 1$, so we have inverses.

g. $\mathbb{Z}$ *with operation* $a \bullet b = a + b + 1$.

Solution: Yes. The operation is associative:

$$(a \bullet b) \bullet c = (a + b + 1) + c + 1 = a + (b + c + 1) + 1 = a \bullet (b \bullet c).$$

The identity element is -1:

$$a \bullet (-1) = a + (-1) + 1 = a,$$

and similarly on the other side. It has inverses:

$$a \bullet (-a - 2) = a + (-a - 2) + 1 = -1,$$

and similarly on the other side.

In fact this group is isomorphic to the additive group of $\mathbb{Z}$, via the map that takes $n \in \mathbb{Z}$ to $n - 1$.

h. $\mathbb{Z}$ *with operation* $a \bullet b = a - b$.

Solution: No. The operation is not associative: $(1-2)-3 = -4$, but $1 - (2 - 3) = 2$.

i. $\mathbb{Q} \setminus \{1\}$ *with operation* $a \bullet b = a + b - ab$. The operation is associative:

$$\begin{aligned}(a \bullet b) \bullet c &= (a + b - ab) + c - (a + b - ab)c \\ &= a + b + c - ab - ac - bc + abc \\ &= a + (b + c - bc) - a(b + c - bc) = a \bullet (b \bullet c).\end{aligned}$$

The identity element is 0:

$$a \bullet 0 = a + 0 - a \cdot 0 = a,$$

and similarly on the other side. It has inverses:

$$a \bullet \frac{a}{a-1} = a + \frac{a}{a-1} - a \cdot \frac{a}{a-1} = \frac{a(a-1) + a - a^2}{a-1} = 0,$$

and similarly on the other side. The quotient $\frac{a}{a-1}$ is ok because $a \neq 1$.

In fact this group is isomorphic to the group of units $\mathbb{Q}^\times = \mathbb{Q} \setminus \{0\}$, via the map that takes $x \in \mathbb{Z}$ to $x + 1$.

§6.1 #4. Let $\mathrm{GL}_2(\mathbb{Z}_2)$ be the group of invertible 2×2 matrices with entries in $\mathbb{Z}_2$. List its elements. What is the order of the group?

Solution: A 2×2 matrix with entries in a field is invertible if and only if its two columns are non-zero and not multiples of one another, and over $\mathbb{Z}_2$, "not multiples of one another" just means "not equal." So after taking the 16 2×2 matrices with entries in $\mathbb{Z}_2$, excluding the four where the two columns are equal, and excluding the six where one column is zero and the other is not, we are left with six elements:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

You may have noticed that these are listed in §6.2 Example 1(f) on page 184.

§6.1 #18. In S_4, let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

Compute...

a. $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$

b. $\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$

c. $\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$

d. $\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$

e. $\sigma^2\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$

f. $\sigma\tau\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

g. $\tau\sigma\tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$