Solutions to Homework 8

1. For cubical dice, the standard way to label the faces is so that opposite faces add up to 7: so 1 is across from 6, 2 is across from 5, and 3 is across from 4. Actually there are two ways to do this, depending on whether 1, 2, and 3 go clockwise or counterclockwise around the vertex that they share.

   But if you relax this requirement, how many ways are there to label the faces of a die?

   Explain what this has to do with the cosets of H in G, where G is the symmetric group S_6 and H is the subgroup given by rotations of a cube.

   Solution: There are $6! = 720$ ways to label the faces of a cube, but the group of rotations has order 24, and if two labelings are related by a rotation then they're related by a *unique* rotation, so there are $720/24 = 30$ ways to label the faces of a die.

   If we choose one labeling and call it the "standard" labeling, then we can get from there to any other labeling by a permutation $\sigma \in G = S_6$, and that labeling is a rotation of the standard one if and only if σ is in the subgroup H. More generally, given a labeling corresponding to any permutation σ, the labelings related to it by rotation are exactly those corresponding to $\rho \circ \sigma$ for some $\rho \in H$. So we see that labelings "up to rotation" are in bijection with right cosets $H\sigma$. (Or if you set things up differently then you can make them be in bijection with left cosets.)

2. Nick, Rachel, Gwen, and Alice (my family) are going to sing a four-part round like "Row, Row, Row Your Boat" or "Frère Jaques." You might say there are $4! = 24$ ways to assign the parts: four choices for who goes first, then three for who goes second, and so on. But after they've been singing for a while, Nick–Rachel–Gwen–Alice will be indistinguishable from Rachel–Gwen–Alice–Nick, for example.

   How many ways are there to assign the parts, up to the notion of equivalence that this suggests?

   Explain what this has to do with cosets of H in G, where G is the symmetric group S_4 and $H \cong \mathbb{Z}_4$ is the cyclic subgroup generated by the permutation

$$\begin{tikzcd} \sigma = 1 \arrow[drrr] & 2 \arrow[dl] & 3 \arrow[dl] & 4 \arrow[dl] \\ 1 & 2 & 3 & 4. \end{tikzcd}$$

   Solution: Once we've forgotten who went first, there are 6 possibilities: you ask who goes after Alice (3 choices), and who goes after them (2 choices), and who goes after them (1 choice).

   If we label the people, perhaps inhumanely, as 1, 2, 3, 4, then any order they can go in corresponds to a permutation $\sigma \in G = S_4$. Let ρ be the 4-cycle shown above, and let $\sigma \in S_4$; then the permutations that are the same as σ once we've forgotten who went first are σ, $\rho \circ \sigma$, $\rho^2 \circ \sigma$, and $\rho^3 \circ \sigma$. This is exactly the right coset $H\sigma$, because $H = \{1, \rho, \rho^2, \rho^3\}$. Thus orderings "up to forgetting who went first" are in bijection with right cosets of H. (Or if you set things up differently then you can make them be in bijection with left cosets.)

3. Let G be a group. Show that the map $\phi\colon G \to G$ given by $\phi(g) = g^2$ is a homomorphism if and only if g is Abelian. (Compare §6.2 #8.)

   Solution: We have $\phi(gh) = ghgh$, and $\phi(g)\phi(h) = gghh$, so ϕ is a homomorphism if and only if

$$ghgh = gghh$$

   for all $g, h \in G$. If G is Abelian then this is true. Conversely, if this is true then we can multiply by g^{-1} on the left and h^{-1} on the right to get $hg = gh$, so G is Abelian.

§6.2 #9. *Show that the map* $\phi\colon \mathbb{C}^\times \to \mathbb{R}^\times$ *given by* $\phi(z) = |z|$ *is a homomorphism. What is* $\ker\phi$*?*

Solution: Because $\mathbb{C}$ is a field, the group of units $\mathbb{C}^\times$ is just $\mathbb{C} \setminus 0$, and similarly $\mathbb{R}^\times = \mathbb{R} \setminus 0$. In both cases the group operation is multiplication, and the identity is 1.

Let $z, w \in \mathbb{C}^\times$ be given, and write $z = a + bi$ and $w = c + di$. Then

$$\begin{aligned}\phi(z) &= |z| = \sqrt{a^2 + b^2},\\ \phi(w) &= |w| = \sqrt{c^2 + d^2},\end{aligned}$$

and

$$\begin{aligned}\phi(zw) &= \phi((ac - bd) + (ad + bc)i)\\ &= \sqrt{(ac - bd)^2 + (ad + bc)^2}\\ &= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2},\end{aligned}$$

where in the last line the cross-terms $-2acbd$ and $+2adbc$ cancelled out. Then we see that

$$\phi(z)\phi(w) = \phi(zw),$$

so ϕ is a homomorphism.

The kernel of ϕ is the set of all $z \in \mathbb{C}^*$ with $|z| = 1$, that is, the unit circle.

§6.2 #10. *Show that the map* $\phi\colon \mathbb{R} \to \mathbb{C}^\times$ *given by* $\phi(t) = \cos(2\pi t) + i\sin(2\pi t)$ *is a homomorphism. What are its kernel and image?*

In $\mathbb{R}$, the group operation is addition, and the identity is 0. In $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, the group operation is multiplication, and the identity is 1.

Let $s, t \in \mathbb{R}$. Using angle addition formulas we find that

$$\begin{aligned}\phi(s + t) &= \cos(2\pi s + 2\pi t) + i\sin(2\pi s + 2\pi t)\\ &= \cos(2\pi s)\cos(2\pi t) - \sin(2\pi s)\sin(2\pi t)\\ &\qquad + i\sin(2\pi s)\cos(2\pi t) + i\cos(2\pi s)\sin(2\pi t).\end{aligned}$$

On the other hand,

$$\phi(s)\phi(t) = \big(\cos(2\pi s) + i\sin(2\pi s)\big)\big(\cos(2\pi t) + i\sin(2\pi t)\big),$$

and when we multiply it out we find that it equals $\phi(s + t)$. Thus ϕ is a homomorphism.

The kernel of ϕ is the set of $t \in \mathbb{R}$ with $\cos(2\pi t) = 1$ and $\sin(2\pi t) = 0$, which is just $\mathbb{Z} \subset \mathbb{R}$. The image of ϕ is the unit circle, which was also the kernel in the previous problem.