Solutions to Homework 9

1. Let G be a group, let $a, b \in G$, and let $H \subset G$ be a subgroup.

   (a) Prove that $aH = H$ if and only if $a \in H$.

   Solution: This follows from part (b) by letting $b = 1$.

   (b) More generally, prove that $aH = bH$ if and only if $b^{-1}a \in H$.

   Solution: Suppose that $aH = bH$. We have $1 \in H$, so $a = a \cdot 1 \in aH$, so $a \in bH$, so $a = bh$ for some $h \in H$, so $b^{-1}a = h$, so $b^{-1}a \in H$.

   Conversely, suppose that $b^{-1}a \in H$. First I claim that $aH \subset bH$. An element of aH can be written as ah for some $h \in H$. Because H is closed under multiplication, we have $(b^{-1}a) \cdot h \in H$. Thus $ah = b \cdot (b^{-1}ah)$ is in bH, so $aH \subset bH$ as claimed. Next, because H is closed under taking inverses, we see that $(b^{-1}a)^{-1} = a^{-1}b$ is in H, so switching the roles of a and b above we find that $bH \subset aH$. Thus $aH = bH$.

§6.3 #13. *Suppose that $H, K \subset G$ are subgroups of orders 5 and 8 respectively. Prove that $H \cap K = \{e\}$.*

We know that $H \cap K$ is a subgroup of H, so $|H \cap K|$ divides $|H| = 5$. It is also a subgroup of K, so $|H \cap K|$ divides $|K| = 8$. Because $\gcd(5, 8) = 1$, we see that $|H \cap K| = 1$, so $H \cap K$ has only one element, namely the identity.

§6.3 #18. *Prove that $H \subset G$ is a normal subgroup if and only if every left coset is a right coset, i.e., $aH = Ha$ for all $a \in G$.*

By the definition on page 192, a subgroup $H \subset G$ is *normal* if for all $a \in G$ and $h \in H$ we have $aha^{-1} \in H$.

First suppose that H is normal, and let $a \in G$. I claim that $aH \subset Ha$: an element of aH can be written as ah for some $h \in H$, and $aha^{-1} \in H$, so $ah = (aha^{-1}) \cdot a$ is in Ha. Similarly we have $Ha \subset aH$, because $ha = a \cdot (a^{-1}ha)$. Thus $aH = Ha$.

Conversely, suppose that $aH = Ha$ for all $a \in G$. Let $a \in G$ and $h \in H$ be given; then $ah \in aH = Ha$, so there is an $h' \in H$ such that $ah = h'a$. Then $aha^{-1} = h'$, so $aha^{-1} \in H$.