Worksheet 2

Math 392, Abstract Algebra

Wednesday, January 6, 2021

Given two ideals $I, J \subset R$, we have defined

$$I \cdot J = \{a_1 b_1 + \cdots + a_k b_k : \\ a_1, \ldots, a_k \in I,\ b_1, \ldots, b_k \in J,\ k \text{ is allowed to vary}\}.$$

We proved in lecture that $I \cdot J$ is an ideal.

Let $R = \mathbb{Z}[\sqrt{-5}]$, and let

$$I = \langle 2, 1 + \sqrt{-5} \rangle$$
$$J = \langle 3, 1 + \sqrt{-5} \rangle$$
$$K = \langle 3, 1 - \sqrt{-5} \rangle$$

1. Prove that every element of $J \cdot K$ is a multiple of 3.

   Conversely, prove that 3 is an element of $J \cdot K$.

   So you've proved that $J \cdot K \subset \langle 3 \rangle$, and by the first problem on Friday's homework, $\langle 3 \rangle \subset J \cdot K$, so $J \cdot K = \langle 3 \rangle$.

2. Prove that every element of $I \cdot I$ is a multiple of 2.

   Conversely, prove that 2 is an element of $I \cdot I$.

   Thus you have proved that $I \cdot I = \langle 2 \rangle$.

3. Challenge: Similarly, prove that $I \cdot J = \langle 1 + \sqrt{-5} \rangle$ and $I \cdot K = \langle 1 - \sqrt{-5} \rangle$.