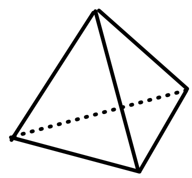
Worksheet 20

Math 392, Abstract Algebra

Friday, March 5, 2021

A regular tetrahedron is a solid with four equilateral triangles for faces, also known as a triangular-based pyramid:



1. Let G be the group of all symmetries of a tetrahedron, including both rotations and reflections. The action of G on the four vertices gives a homomorphism
$$\phi \colon G \to S_4,$$
and it's injective. Discuss any questions you have about what this means or why it's true.

2. For each pair of vertices, there is a reflection $g \in G$ that switches those two and fixes the other two. Draw a little picture of this, including the plane that you're reflecting across.

3. Thus the image of ϕ includes all transpositions, and every element of S_4 is a product of transpositions, so ϕ is surjective, so ϕ is an isomorphism. Can you fill in the details of this to give a complete proof?

   Here's a start: Given an element $\sigma \in S_4$, we can choose transpositions $\tau_1, \tau_2, \ldots, \tau_k$ such that $\sigma = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_k$. There are $g_1, g_2, \ldots, g_k \in G$ such that $\phi(g_i) = \tau_i$. Then...

4. Draw a picture of the rotation $g \in G$ with $\phi(g) = (1\,2\,3)$, including the axis that you're rotating around.

5. Draw a picture of the rotation $g \in G$ with $\phi(g) = (1\,2)(3\,4)$, including the axis that you're rotating around.

6. Challenge: Think about the element $g \in G$ with $\phi(g) = (1\,2\,3\,4)$.