Worksheet 21

Math 392, Abstract Algebra

Monday, March 8, 2021

Let $G = \mathrm{GL}_2(\mathbb{Z}_2)$, the group of invertible 2×2 matrices with entries in $\mathbb{Z}_2$. On Homework 6 you found that it has six elements:

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}.$$

Thus by our work in lecture today, G is either isomorphic to $\mathbb{Z}_6$ or to S_3.

1. Show that G is not Abelian by finding two matrices $A, B \in G$ such that $AB \neq BA$. (Remember that all the arithmetic is done mod 2.)

So G must be isomorphic to S_3. Here we will see the isomorphism explicitly by letting G act on a set with three elements.

Let V be the set of column vectors with entries in $\mathbb{Z}_2$:

$$V = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Then G acts on V in the way you expect: for $A \in G$ and $v \in V$, we get $A \cdot v \in V$ by matrix multiplication.

2. Convince yourselves that the zero vector $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ is in an orbit by itself. So G permutes the other three vectors, and if you label them 1, 2, 3 in any way you like, then you get a homomorphism $\phi\colon G \to S_3$.

3. Which matrices in G map to the transpositions $(1\,2)$, $(1\,3)$, and $(2\,3)$ in S_3? Which matrices map to the 3-cycles $(1\,2\,3)$ and $(1\,3\,2)$?

So you see that ϕ is a bijection, so it's an isomorphism.