Worksheet 3

Math 392, Abstract Algebra

Friday, January 8, 2021

In lecture we saw that $I \cdot J$ is contained in $I \cap J$, and if $I + J = \langle 1 \rangle$ then they're equal, which generalized the fact that $\operatorname{lcm}(m, n)$ divides mn, and if $\gcd(m, n) = 1$ then they're equal.

The converse statement would be "if $I \cdot J = I \cap J$ then $I + J = \langle 1 \rangle$." This is true in some rings, including $\mathbb{Z}$ and $\mathbb{Z}[\sqrt{-5}]$ and $\mathbb{Q}[x]$, but in general it's not true. Here you'll work out an example.

1. Let $R = \mathbb{Z}[x]$, the ring of polynomials with integer coefficients. Let $I = \langle 2 \rangle$. Convince yourselves that I is the set of polynomials whose coefficients are even.

2. Let $J = \langle x \rangle$. Convince yourselves that J is the set of polynomials whose constant term is zero.

3. From Wednesday's lecture we know that $I \cdot J = \langle 2x \rangle$. On the other hand, we can see that $I \cap J$ is the set of polynomials whose coefficients are even *and* whose constant term is zero. Convince yourselves that these are the same set.

4. From Wednesday's lecture we know that $I + J = \langle 2, x \rangle$. Convince yourselves that this is the set of polynomials whose constant term is even. Thus $I + J \neq \langle 1 \rangle$, because it doesn't contain 1.

5. Challenge: Prove that $\langle 2, x \rangle$ is not a principal ideal: that is, there is no $f \in \mathbb{Z}[x]$ such that $\langle 2, x \rangle = \langle f \rangle$.