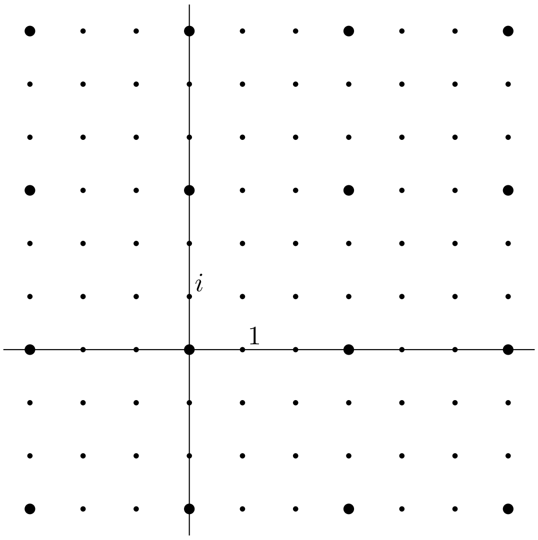
Worksheet 4

Math 392, Abstract Algebra

Monday, January 11, 2021

Here's a picture of $R = \mathbb{Z}[i]$, with the ideal $I = \langle 3 \rangle$ shown in heavier dots.



1. Convince yourselves that R/I has nine elements:

$$R/I = \{\bar{0}, \bar{1}, \bar{2}, \bar{i}, \overline{1+i}, \overline{2+i}, \overline{2i}, \overline{1+2i}, \overline{2+2i}\}$$

   So there are two things to see: First, for any $z \in R$, the equivalence class $\bar{z} \in R/I$ is equal to one of those nine. And second, none of those nine are equal to each other.

2. I claim that R/I is a field: that is, for every $\bar{z} \in R/I$ with $\bar{z} \neq \bar{0}$, there is a $\bar{w} \in R/I$ such that $\bar{z} \cdot \bar{w} = \bar{1}$. For example, $\bar{1} \cdot \bar{1} = \bar{1}$, and $\bar{2} \cdot \bar{2} = \bar{1}$. Find the multiplicative inverses of $\bar{i}$, $\overline{i+1}$, and so on.

3. Challenge: But if $J = \langle 5 \rangle \subset R$ then R/J is not a field, because it has zero divisors. Hint: look at $\overline{2+i}$ and $\overline{2-i}$.