Worksheet 6

Math 392, Abstract Algebra

Friday, January 15, 2021

Definition from lecture: an ideal $P \subseteq R$ is called prime if $rs \in P$ implies $r \in P$ or $s \in P$.

Let $R = \mathbb{Z}[x]$ and let $P = \langle 2, x \rangle$. On worksheet 3 you saw that $P$ is the set of polynomials whose constant term is even.

1. Let $\phi: R \to \mathbb{Z}_2$ be the map given by
   \[
   \phi(a_n x^n + \cdots + a_1 x + a_0) = \bar{a}_0.
   \]
   Convince yourselves that $\phi$ is a homomorphism.

2. Convince yourselves that $\ker \phi = P$.
   (That is, for all $f \in R$ we have $\phi(f) = 0$ if and only if $f \in P$.)

3. Use the previous problem to prove that $P$ is prime.

4. Challenge: Prove that the ideal $I = \langle 2, x^2 + 5 \rangle \subseteq R$ is not prime, by showing that $(x + 1)(x - 1) \in I$ but $x + 1 \notin I$ and $x - 1 \notin I$. 