Worksheet 7

Math 392, Abstract Algebra

Wednesday, January 20, 2021

Based on §4.3 #10.

Noether's first isomorphism theorem states that from any homomorphism $\phi\colon R \to S$ we get an isomorphism $R/\ker\phi \cong \operatorname{im}\phi$. In particular, if ϕ is surjective then $R/\ker\phi \cong S$

1. Prove that $\mathbb{Z}[i]/\langle 1+i\rangle \cong \mathbb{Z}_2$. By the first isomorphism, it is enough to find a surjective homomorphism $\phi\colon \mathbb{Z}[i] \to \mathbb{Z}_2$ such that $\ker\phi = \langle 1+i\rangle$.

   Hint: You met such a homomorphism on Worksheet 23 (which accompanied Lecture 26) last quarter.

2. Prove that $\mathbb{Z}[i]/\langle 3+i\rangle \cong \mathbb{Z}_{10}$, again by finding a surjective homomorphism $\phi\colon \mathbb{Z}[i] \to \mathbb{Z}_{10}$ such that $\ker\phi = \langle 3+i\rangle$.

   You could try $\phi(a+bi) = \bar{a} + \bar{b}\bar{n}$ for various $\bar{n} \in \mathbb{Z}_{10}$.

   If you want ϕ to be a homomorphism then you at least need $\bar{n}^2 = -\bar{1}$ in $\mathbb{Z}_{10}$. (Do you see why this is necessary?)

   If you want $\ker\phi$ to be $\langle 3+i\rangle$ then you at least need $\phi(3+i) = \bar{0}$. (Do you see why this is necessary?)

   So that should narrow down your choice of $\bar{n}$.

3. Prove that $\mathbb{Z}[i]/\langle 2+3i\rangle \cong \mathbb{Z}_{13}$ using the same approach.

4. Challenge: Prove that the ring $\mathbb{Z}[i]/\langle 2\rangle$ has four elements, but it is not isomorphic to $\mathbb{Z}_4$.