Based on §4.3 #10.

Noether’s first isomorphism theorem states that from any homomorphism \( \phi: R \to S \) we get an isomorphism \( R/\ker \phi \cong \text{im } \phi \). In particular, if \( \phi \) is surjective then \( R/\ker \phi \cong S \).

1. Prove that \( \mathbb{Z}[i]/\langle 1 + i \rangle \cong \mathbb{Z}_2 \). By the first isomorphism, it is enough to find a surjective homomorphism \( \phi: \mathbb{Z}[i] \to \mathbb{Z}_2 \) such that \( \ker \phi = \langle 1 + i \rangle \).
   
   Hint: You met such a homomorphism on Worksheet 23 (which accompanied Lecture 26) last quarter.

2. Prove that \( \mathbb{Z}[i]/\langle 3 + i \rangle \cong \mathbb{Z}_{10} \), again by finding a surjective homomorphism \( \phi: \mathbb{Z}[i] \to \mathbb{Z}_{10} \) such that \( \ker \phi = \langle 3 + i \rangle \).
   
   You could try \( \phi(a + bi) = \bar{a} + \bar{b}\bar{n} \) for various \( \bar{n} \in \mathbb{Z}_{10} \).
   
   If you want \( \phi \) to be a homomorphism then you at least need \( \bar{n}^2 = -1 \) in \( \mathbb{Z}_{10} \). (Do you see why this is necessary?)
   
   If you want \( \ker \phi \) to be \( \langle 3 + i \rangle \) then you at least need \( \phi(3 + i) = 0 \). (Do you see why this is necessary?)
   
   So that should narrow down your choice of \( \bar{n} \).

3. Prove that \( \mathbb{Z}[i]/\langle 2 + 3i \rangle \cong \mathbb{Z}_{13} \) using the same approach.

4. Challenge: Prove that the ring \( \mathbb{Z}[i]/\langle 2 \rangle \) has four elements, but it is not isomorphic to \( \mathbb{Z}_4 \).