Final Exam

Tuesday, December 5, 2023

There are 57 points in total.

1. Let X and Y be topological spaces, and let $f\colon X \to Y$.

   (a) (5 points) Suppose we can write $X = U_1 \cup \cdots \cup U_n$, where each U_i is open. Let $f_i\colon U_i \to Y$ be the restriction of f to U_i; that is, for all $p \in U_i$, we set $f_i(p) = f(p)$. Prove that f is continuous if and only if each f_i is continuous.

   (b) (3 points) Suppose we can write $X = F_1 \cup \cdots \cup F_n$, where each F_i is closed. Let $f_i\colon F_i \to Y$ is the restriction of f to F_i. Prove that f is continuous if and only if each f_i is continuous.

   (c) (5 points) In (a) and (b), if we replace the finite union of open or closed sets with a countable union, then one statement remains true and one becomes false. Give a counterexample to the false one.

   (d) (5 points) Let $f, g\colon X \to \mathbb{R}$ be continuous maps (with in the usual topology on $\mathbb{R}$). Prove that $\max(f, g)$ is continuous.
   Hint: Apply part (b) with two sets, one of which is

   $$F = \{p \in X : f(p) \geq g(p)\}.$$

   To prove that F is closed, you can use the fact that $f - g$ is continuous, which we proved in lecture.

2. True or false: (2 points each, no proofs or counterexamples required)

   (a) A closed subset of a Hausdorff space is compact.

   (b) A compact subset of a Hausdorff space is closed.

   (c) A closed subset of a compact space is compact.

   (d) A compact subset of a compact space is closed.

   (e) The continuous image of a Hausdorff space is Hausdorff.

   (f) The continuous image of a compact space is compact.

3. (a) (3 points) Define an *open cover* of a subset $A \subset X$ and a *subcover*. Define what it means for A to be *compact*.

   (b) (5 points) Let X be a metric space, let $A \subset X$, and suppose there is an $r > 0$ such that for all $p, q \in A$ with $p \neq q$ we have $d(p, q) > r$. Prove that A is closed.

   Hint: You might prove that any sequence in A that converges to a limit in X is eventually constant, so the limit is in A. Or you might prove that $X \setminus A$ is open, starting by proving that for any $x \in X \setminus A$, the ball $B_{r/2}(x)$ contains at most one point of A. Or you might have another idea.

   (c) (5 points) Prove that if A is compact then A is finite.

   Hint: Consider the open cover $\{B_r(p) : p \in A\}$.

   (d) (3 points) Prove that if X is compact then A is finite.

   (e) (3 points) Give an example of a non-compact metric space X and an infinite subset A with the property from part (b).

4. Let $X = [0, 2\pi)$ and let $Y = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be the unit circle, both in the usual topology.

   (a) (3 points) Write down a continuous bijection $f : X \to Y$. (No proof required.)

   (b) (5 points) Prove that there is no continuous bijection $g : Y \to X$.