Homework 1

Due Monday, October 2, 2023

You may work with other students on these problems, but you must write them up yourself, in your own words. If you write by hand, use pencil, because you will inevitably want to erase something. If you type, use TEX, not Microsoft Word. Do not use the symbols $\forall$ or $\exists$ or $\therefore$. You have time to write "for every" or "there is" or "therefore" (or better yet, "so").

1. In lecture we introduced three standard metrics on $\mathbb{R}^2$: the Euclidean metric
$$d_2((x_1, y_1),\, (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$
the taxicab metric
$$d_1((x_1, y_1),\, (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|,$$
and the square metric
$$d_\infty((x_1, y_1),\, (x_2, y_2)) = \max\left\{|x_1 - x_2|, |y_1 - y_2|\right\}.$$
   (a) For each of the three metrics, sketch the open ball of some radius $r > 0$ around the origin
$$B_r(0) = \{p \in \mathbb{R}^2 : d(p, 0) < r\}.$$
   (b) For one of the three metrics (your choice), prove or give a counterexample to the following statement: a sequence of points $p_n = (x_n, y_n)$ converges to a limit $p = (x, y)$ if and only if $x_n \to x$ and $y_n \to y$ separately, as sequences in $\mathbb{R}$ with the usual metric.

2. Consider the following silly metric on $\mathbb{R}^2$:

$$d((x_1, y_1),\ (x_2, y_2)) = \begin{cases} |y_1 - y_2| & \text{if } x_1 = x_2 \\ |y_1 - y_2| + 1 & \text{if } x_1 \neq x_2. \end{cases}$$

   (a) Prove that d is a metric, that is, it satisfies the three axioms.

   (b) Sketch the open balls of radius $1/2$, 1, and 2 around the origin in this metric.

   (c) Give an example of a sequence that converges in the Euclidean metric d_2 but not in our silly metric d.

   (d) Show that every sequence that converges in d also converges d_2.

3. Let (X, d_X) and (Y, d_Y) be metric spaces, let $\{p_n\}$ be a sequence in X that converges to a point $\ell \in X$, and let $f\colon X \to Y$ be continuous at ℓ. Show that the sequence $\{f(p_n)\}$ in Y converges to $f(\ell)$.

4. Optional: In lecture we saw an example of a sequence in $C([0,1])$ that converges in the L^1 metric but not in the sup metric. Show that the reverse cannot happen: every sequence that converges in the sup metric converges in the L^1 metric.

5. What is one question you have about last week's lectures?