Homework 2

Due Monday, October 9, 2023

1. Let (X, d_X) and (Y, d_Y) and (Z, d_Z) be metric spaces. Let $f\colon X \to Y$ be continuous at a point $p \in X$, and let $g\colon Y \to Z$ be continuous at $f(p)$. Prove that $g \circ f$ is continuous at p.

2. Let X be any set, and let d_X be the *discrete metric*

$$d_X(p,q) = \begin{cases} 0 & \text{if } p = q, \text{ or} \\ 1 & \text{if } p \neq q. \end{cases}$$

   (a) Prove that d_X is a metric.

   (b) Let (Y, d_Y) be another metric space (not necessarily discrete). Prove that every map $f\colon X \to Y$ is continuous.

3. Let (X, d) be a metric space. Prove the *reverse triangle inequality*:

$$|d(p,q) - d(p,r)| \leq d(q,r).$$

   Include an appropriate picture.

4. Let $f_n\colon [0,2] \to \mathbb{R}$ be the piecewise function that goes from $(0,0)$ to $(1-\frac{1}{n}, 0)$ to $(1+\frac{1}{n}, 1)$ to $(2,1)$. Sketch the graph of f_n, and of f_m for some $m \neq n$. Prove that this sequence is Cauchy in the L^1 metric, which is defined as

$$d_1(f,g) = \int_0^2 |f(x) - g(x)|\, dx.$$

5. Let $X = \mathbb{R}^2$ with the Euclidean metric. Sketch the subset

$$A = \{(x,y) \in \mathbb{R}^2 : x \neq 0 \text{ or } y = 0\}.$$

   Prove that A is neither open nor closed.

6. Optional: In lecture, when we discussed the construction of the completion a metric space (X, d), there was a laundry list of things to prove, and we proved some of them. Here's one that we skipped: if $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in X, then the limit

$$\lim_{n\to\infty} d(p_n, q_n)$$

   exists. Prove this by showing that $d(p_n, q_n)$ is a Cauchy sequence in $\mathbb{R}$ with its usual metric. You can use the fact that $\mathbb{R}$ is complete without proving it.

   Hint: The reverse triangle inequality may be useful.

7. What is one question you have about last week's lectures?