Homework 3

Due Monday, October 16, 2023

1. In lecture we proved some facts about unions and intersections of closed sets. We also proved that a set is open if and only if its complement is closed, so those facts are equivalent to the following facts about intersections and unions of open sets by de Morgan's laws. But prove the following facts directly. Let (X, d) be a metric space.

   (a) Let $U, V \subset X$ be two open sets. Prove that $U \cap V$ is open.

   (b) Give an example of countably many open sets $U_1, U_2, \ldots \subset X$ such that their intersection $U_1 \cap U_2 \cap \cdots$ is not open.

   (c) Let $\mathcal{U}$ be a collection of open sets in X. Prove that the union $\bigcup \mathcal{U}$ is open. (By definition, $\bigcup \mathcal{U} = \{p \in X : p \in U \text{ for some } U \in \mathcal{U}\}$.)

2. Let (X, d) be a metric space. For a subset $A \subset X$, we defined the *closure* $\bar{A}$ as the set of points $p \in X$ for which there a sequence $a_1, a_2, \ldots$ in A that congerges to p, and proved that this is equivalent to saying that for every $\epsilon > 0$, the set $B_\epsilon(p) \cap A$ is not empty.

   (a) Prove that $A \subset \bar{A}$.

   (b) Prove that $\bar{A}$ is closed.

   Hint: It's easier to use the characterization of $\bar{A}$ involving balls than the one involving sequences.

   (c) Prove that $\bar{A}$ is the smallest closed set containing A, in the following sense: if $F \subset X$ is closed and $A \subset F$, then $\bar{A} \subset F$.

   (d) Prove that if $A \subset B$ then $\bar{A} \subset \bar{B}$.

   Hint: Use (b) and (c), and don't work hard.

   (e) Prove that $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

   Hint: Use (b), (c), (d), and the fact that a union of two closed sets is closed.

   (f) Prove that $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$. Give an example where the inclusion is strict.

3. Take $\mathbb{R}$ and $\mathbb{R}^2$ with their Euclidean metrics, and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   (a) Argue that for each $c \in \mathbb{R}$, the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = f(x,c)$ is continuous. You may assert without proof the fact that a quotient of two polynomials is continuous when the denominator is not equal to zero.

   (b) Write "Similarly, for each $c \in \mathbb{R}$, the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(y) = f(c,y)$ is continuous."

   (c) Prove that nonetheless that f is not continuous.
   Hint: Consider the sequence $(\frac{1}{n}, \frac{1}{n}) \to (0,0)$.

   The point of this problem is that saying a function of several variables is continuous is stronger than just saying it's continuous "one variable at a time."

4. Optional: Let

$$W = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots, 0\}$$

   with the usual metric inherited from $\mathbb{R}$. Let X be another metric space. Given a sequence $p_1, p_2, \ldots \in X$ and a point $\ell \in X$, show that the function $f\colon W \to X$ defined by

$$\begin{cases} f(\frac{1}{n}) = p_n, \\ f(0) = \ell \end{cases}$$

   is continuous if and only if $p_n \to \ell$.

5. What is one question you have about last week's lectures?