Homework 4

Due Monday, October 23, 2023

1. On homework 1 we saw three different metrics on $\mathbb{R}^2$. Prove one of the following:

   (a) A subset $A \subset \mathbb{R}^2$ is open in the Euclidean metric if and only if it is open in the taxicab metric.

   (b) A subset $A \subset \mathbb{R}^2$ is open in the Euclidean metric if and only if it is open in the square metric.

   (c) A subset $A \subset \mathbb{R}^2$ is open in the taxicab metric if and only if it is open in the square metric.

2. Let $f\colon X \to Y$ be any map of sets. The *image* of a subset $A \subset X$ is

$$f(A) = \{f(a) : a \in A\},$$

   or equivalently

$$f(A) = \{y \in Y : y = f(a) \text{ for some } a \in A\}.$$

   This is a subset of Y.

   (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Find $f(A)$ for the following subsets $A \subset \mathbb{R}$: the intervals $[-1, 1]$, $[-1, 1)$, $(-1, 1)$, $[0, 1]$, $[0, 1)$, and $(0, 1)$, and the singletons $\{-1\}$, $\{0\}$, and $\{1\}$.

   (b) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x$. Choose a few subsets $A \subset \mathbb{R}^2$ and sketch both A and $f(A)$.

   (c) Now let $f\colon X \to Y$ be arbitrary, and let $A, B \subset X$. Prove that if $A \subset B$ then $f(A) \subset f(B)$. Prove that $f(A \cup B) = f(A) \cup f(B)$. Prove that $f(A \cap B) \subset f(A) \cap f(B)$, but give an example where the two are not equal.

3. Let $f\colon X \to Y$ be a map of sets. The *preimage* of a subset $B \subset Y$ is

$$f^{-1}(B) = \{x \in X : f(x) \in B\},$$

or equivalently

$$f^{-1}(B) = \{x \in X : f(x) = b \text{ for some } b \in B\}.$$

This is a subset of X.

(a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Find $f^{-1}(B)$ for the following subsets $B \subset \mathbb{R}$: the intervals $[-1, 1]$, $[-1, 1)$, $(-1, 1)$, $[0, 1]$, $[0, 1)$, and $(0, 1)$, and the singletons $\{-1\}$, $\{0\}$, and $\{1\}$.

(b) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = xy$. Sketch $f^{-1}(B)$ when B is $\{0\}$, $\{1\}$, $[0, 1]$, $(0, 1)$, and $[0, 1)$.

(c) Now let $f\colon X \to Y$ be arbitrary, and let $A, B \subset Y$. Prove that if $A \subset B$ then $f^{-1}(A) \subset f^{-1}(B)$. Prove that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, and that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

4. (a) The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x + 1 & \text{if } x > 0 \end{cases}$$

is discontinuous (with the usual metric on $\mathbb{R}$). Give an example of an open set $V \subset \mathbb{R}$ such that $f^{-1}(V)$ is not open.

(b) Optional: Last week you proved that he function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is discontinuous (with the usual metric on $\mathbb{R}^2$ and $\mathbb{R}$). Give an example of an open set $V \subset \mathbb{R}$ such that $f^{-1}(V)$ is not open.

5. What is one question you have about last week's lectures?