Homework 5

Due Monday, October 30, 2023

We're going easy this week because of the exam.

1. Let X be a set. A *topology* on X is a set T of subsets of X, such that

   (i) The empty set $\varnothing$ and the whole set X belong to T.

   (ii) The intersection of finitely many members of T belongs to T: if $U_1, U_2, \ldots U_n \in T$ then $U_1 \cap U_2 \cap \cdots \cap U_n \in T$.

   (iii) The union of arbitrarily many members of T belongs to T: if $S \subset T$ then $\bigcup S \in T$.

   Recall from problem 1 on homework 3 that

   $$\bigcup S = \{p \in X : p \in U \text{ for some } U \in S\}.$$

   For example, if we choose a metric d on X and let T be the set of subsets of X that are open with respect to d, then problem 1 from homework 3 shows that T is a topology.

   (a) Let T be the set of subsets $U \subset \mathbb{R}$ that such that $\mathbb{R} \setminus U$ is finite, together with the empty set. Prove that T is a topology. (It is called the "finite complement topology.")

   (b) Let T be the set of subsets $U \subset \mathbb{R}$ such that U contains 0, together with the empty set. Prove that T is a topology.

   (c) Let T be the subsets of $\mathbb{R}$ of the form (a, ∞) for some $a \in \mathbb{R}$, together with the empty set and the whole set $\mathbb{R}$. Prove that T is a topology. (It is called the "lower semi-continuous topology.")

2. Let $f\colon X \to Y$, let $A \subset X$, and let $B \subset Y$.

   (a) Show that $A \subset f^{-1}(B)$ if and only if $f(A) \subset B$.

   (b) Show that $A \subset f^{-1}(f(A))$. Show that equality holds if f is injective. Give an example to show that equality need not hold in general.

   (c) Show that $f(f^{-1}(B)) \subset B$. Show that equality holds if f is surjective. Give an example to show that equality need not hold in general.

3. Optional, due in two weeks (11/13): Read "The emergence of open sets, closed sets, and limit points in analysis and topology" by Gregory H. Moore, which is linked on Canvas and on the course web page.

   (a) What is one thing you read that confused you?

   (b) What is one thing you read that surprised you?

4. What is one question you have about last week's lectures?