Homework 6

Due Monday, November 6, 2023

1. In problem 1 last week we introduced three unusual topologies on $\mathbb{R}$.

   (a) Find the closure, interior, and boundary of $\mathbb{Z}$ as a subset of $\mathbb{R}$ with the finite complement topology from 1(a) last week.

   (b) Find the closure, interior, and boundary of the one-point subsets $\{0\}$ and $\{1\}$ as subsets of $\mathbb{R}$ with the "particular point" topology from 1(b) last week.

   (c) Find the closure, interior, and boundary of the interval $(0, 1)$ as a subset of $\mathbb{R}$ with the lower semi-continuous topology from 1(c) last week.

2. Let X be a topological space. Let $Y \subset X$, and give Y the subspace topology. Let $A \subset Y$.

   (a) Show that if A is closed in Y and Y is closed in X, then A is closed in X.

   (b) Give two examples to show that if A is closed in Y and Y is *not* closed in X, then A may or may not be closed in X.

   (c) Show that if A is open in Y and Y is open in X, then A is open in X.

   (d) Give two examples to show that if A is open in Y and Y is *not* open in X, then A may or may not be open in X.

   (e) Let $A \subset Y$, let $\mathrm{cl}_X(A)$ denote the closure of A in X, and let $\mathrm{cl}_Y(A)$ denote the closure of A in Y. Show that

$$\mathrm{cl}_Y(A) = \mathrm{cl}_X(A) \cap Y.$$

(f) Let $A \subset Y$, let $\mathrm{int}_X(A)$ denote the interior of A as a subset of X, and let $\mathrm{int}_Y(A)$ denote the interior of A as a subset of Y. Show that

$$\mathrm{int}_X(A) \subset \mathrm{int}_Y(A).$$

(g) Give an example where the inclusion in part (f) is strict.

3. (a) Let X be a topological space, and suppose that we can write $X = F_1 \cup \ldots \cup F_n$, where each F_i is closed. Let Y be another topological space, let $f\colon X \to Y$, and let $f_i\colon F_i \to Y$ be the restriction of f to F_i: that is, for $x \in F_i$ we set $f_i(x) = f(x)$.

   Prove that f is continuous if and only if f_i is continuous for all i.

   Hint: Use the fact that a map is continuous if and only if the preimage of every closed set is closed.

   (b) This is usually applied to show that a piecewise function is continuous. Consider the function $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 1/3, \\ 3x - 1 & \text{if } 1/3 \leq x \leq 2/3, \\ 1 & \text{if } x \geq 2/3. \end{cases}$$

   If we wanted to apply part (a) to show that f is continuous, what should sets should we take for the F_i?

4. Optional, due next week (11/13): Read “The emergence of open sets, closed sets, and limit points in analysis and topology” by Gregory H. Moore, which is linked on Canvas and on the course web page.

   (a) What is one thing you read that confused you?

   (b) What is one thing you read that surprised you?

5. What is one question you have about last week’s lectures?