Homework 7

Due Monday, November 13, 2023

1. Let (X, d) be a metric space. A function $f\colon X \to \mathbb{R}$ is called *lower semi-continuous* at a point $p \in X$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that $d(p, q) < \delta$ implies $f(q) > f(p) - \epsilon$. The idea is that in the limit, f can only jump down. You can guess what *upper semi-continuous* means.

   (a) Let $X = \mathbb{R}$ with the usual metric, and consider the floor function $\lfloor x \rfloor$, which returns the greatest integer $\leq x$, and the ceiling function $\lceil x \rceil$, which returns the least integer $\geq x$. Which one is lower semi-continuous, and which one is upper semi-continuous?

   (You don't have to prove it.)

   (b) Prove that $f\colon X \to \mathbb{R}$ is lower semi-continuous (at every point) if and only if it is continuous as a map of topological spaces when the codomain $\mathbb{R}$ is given the lower semi-continuous topology defined in problem 1(c) of homework 5.

2. Let (X, d) be a metric space. A function $f\colon \mathbb{R} \to X$ is called *continuous from the right* at a point $p \in \mathbb{R}$ if $\lim_{q \to p^+} f(q) = f(p)$: that is, for every $\epsilon > 0$ there is a $\delta > 0$ such that $p \leq q < p + \delta$ implies $d(f(p), f(q)) < \epsilon$. You can guess what *continuous from the left* means.

   (a) Again take $X = \mathbb{R}$ in the usual metric, and consider the floor and ceiling functions. Which one is continuous from the right, and which one is continuous from the left?

   (You don't have to prove it.)

   (b) Prove that a map $f\colon \mathbb{R} \to X$ is continuous from the right (at every point) if and only if it is continuous as a map of topological spaces when the domain $\mathbb{R}$ is given the *lower limit topology*, which is defined by saying that a set is open if and only if it's a union of intervals of the form $[a, b)$.

3. Given a map $f\colon X \to Y$, we can consider its graph

$$\Gamma_f = \{(x, y) \in X \times Y : y = f(x)\}.$$

   (a) Prove that if X and Y are topological spaces, Y is Hausdorff, and f is continuous, then Γ_f is closed.

   Hint: You could do this by hand, or you could consider the preimage of the diagonal $\Delta \subset Y \times Y$ and under the map $X \times Y \to Y \times Y$ that sends (x, y) to $(f(x), y)$.

   (b) Give an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ that is not continuous (in the usual topology) but whose graph is nonetheless closed.

   Hint: It won't work if f is bounded.

4. Optional:

   (a) Let X and Y be topological spaces, and suppose that Y is Hausdorff. Prove that if two continuous maps $f, g\colon X \to Y$ agree on a dense subset $D \subset X$, then $f = g$.

   Hint: Let $E = \{x \in X : f(x) = g(x)\}$, and prove that it's closed.

   (b) Give a counterexample when Y is not Hausdorff.