Homework 8

Due Monday, November 20, 2023

1. Let X be a topological space. In lecture we defined what it means for a subset $A \subset X$ to be compact. Prove that A is compact as a subset of X if and only if A is compact as a subset of itself (in the subspace topology).

2. Let $X = \mathbb{R}$ with the usual topology, and let $A = \mathbb{Q} \cap [0, 1]$. Give an example of an open cover of A with no finite subcover.

3. Let X be a Hausdorff space. In lecture we proved that if $A \subset X$ is compact and $p \in X \setminus A$, then there are disjoint open sets $U, V \subset X$ with $A \subset U$ and $p \in V$. Now prove that if $A, B \subset X$ are disjoint compact sets then there are disjoint open sets $U, V \subset X$ with $A \subset U$ and $B \subset V$.

4. Optional, but encouaged: If (X, d) is a metric space and $A, B \subset X$, we can define
$$d(A, B) = \inf_{a \in A,\, b \in B} d(a, b).$$
Of course if A and B intersect then $d(A, B) = 0$, but...
   (a) In $\mathbb{R}^2$ with the usual metric, give an example of an open set U and a point $p \notin U$ with $d(U, \{p\}) = 0$.
   (b) In $\mathbb{R}^2$ with the usual metric, give an example of two disjoint closed sets F and G with $d(F, G) = 0$.
   (c) In general, prove that if F is closed and $p \notin F$ then $d(F, \{p\}) > 0$.
   (d) Prove that if F is closed, K is compact, and $F \cap K = \varnothing$, then $d(F, K) > 0$.