# Homework 9

Due Friday, December 1, 2023

1. This week we proved that a closed, bounded subset of $\mathbb{R}^n$ (with the usual metric) is compact, but this is not true in every complete metric space. Let $X = C([0,1])$ with the sup metric, and let $B \subset X$ be the closed ball of radius 1 around the origin. Prove that $B$ is not compact by constructing a sequence $f_1, f_2, \ldots$ in $B$ that has no convergent subsequence.

2. In week 5 we proved the following generalization of the nested interval theorem: If $X$ is a complete metric space and $F_1 \supset F_2 \supset \cdots$ is a nested sequence of closed subsets whose diameters tend to zero, then the intersection $F_1 \cap F_2 \cap \cdots$ is not empty.

   Now prove another generalization: If $X$ be a compact topological space and $F_1 \supset F_2 \supset \cdots$ is a nested sequence of closed sets, then the intersection $F_1 \cap F_2 \cap \cdots$ is not empty.

   Hint: Otherwise the complements $X \setminus F_1$, $X \setminus F_2$, $\ldots$ form an open cover of $X$.

3. Optional: Let $X$ is a metric space. A map $f \colon X \to X$ is called a *contraction mapping* if there is an $r \in [0, 1)$ such that for all $p, q \in X$ we have $d(f(p), f(q)) \leq r \cdot d(p, q)$. In week 3 we proved that if $X$ is complete, then any contraction mapping has a fixed point.

   (a) A map $f \colon X \to X$ is called a *weak contraction mapping* if for all $p \neq q$ we have $d(f(p), f(q)) < d(p, q)$. Prove that if $X$ is compact space, then any weak contraction mapping has a fixed point.

   Hint: Apply the extreme value theorem to the function $F \colon X \to \mathbb{R}$ given by $F(p) = d(p, f(p))$, which you'll need to argue is continuous.

   (b) Let $X = [1, \infty)$ with the usual metric. Prove that the map $f \colon X \to X$ given by $f(x) = x + \frac{1}{x}$ is a weak contraction mapping with no fixed point.